Lesson 4:
Theory of cosmological perturbations

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Early universe cosmology, USP
29/Nov/2022

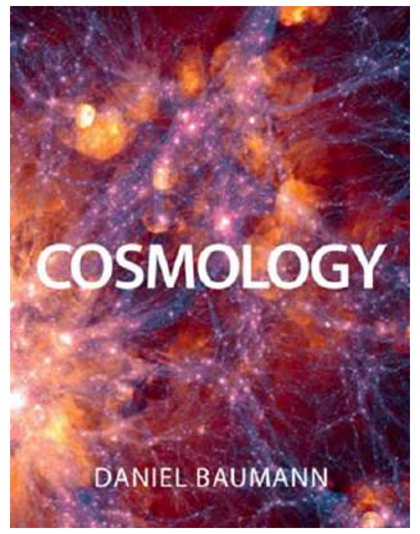
Image: Nasa,ESA, CSA, and STScI

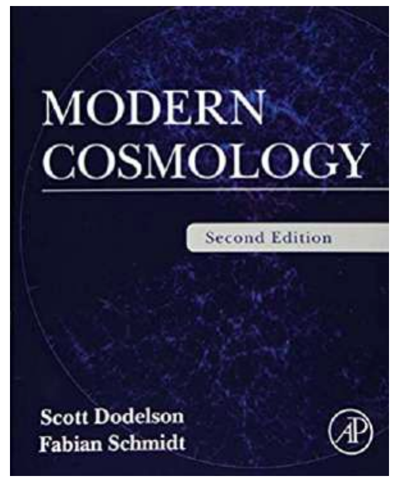
Early universe cosmology

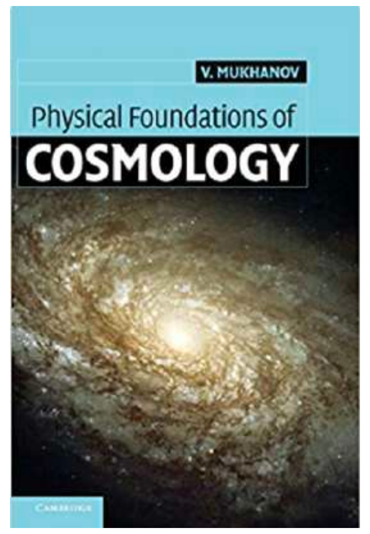
References:

- Daniel Baumann, Cosmology, Cambridge University Press, 2022.
- Viatcheslav Mukhanov, Physical Foundations of Cosmology, Cambridge University Press, 2005

- Daniel Baumann, TASI lectures on inflation
- (Recurso em português) Tese de mestrado Elisa G. M. Ferreira (capítulos 2 e 3)







Questions: elisa@if.usp.br

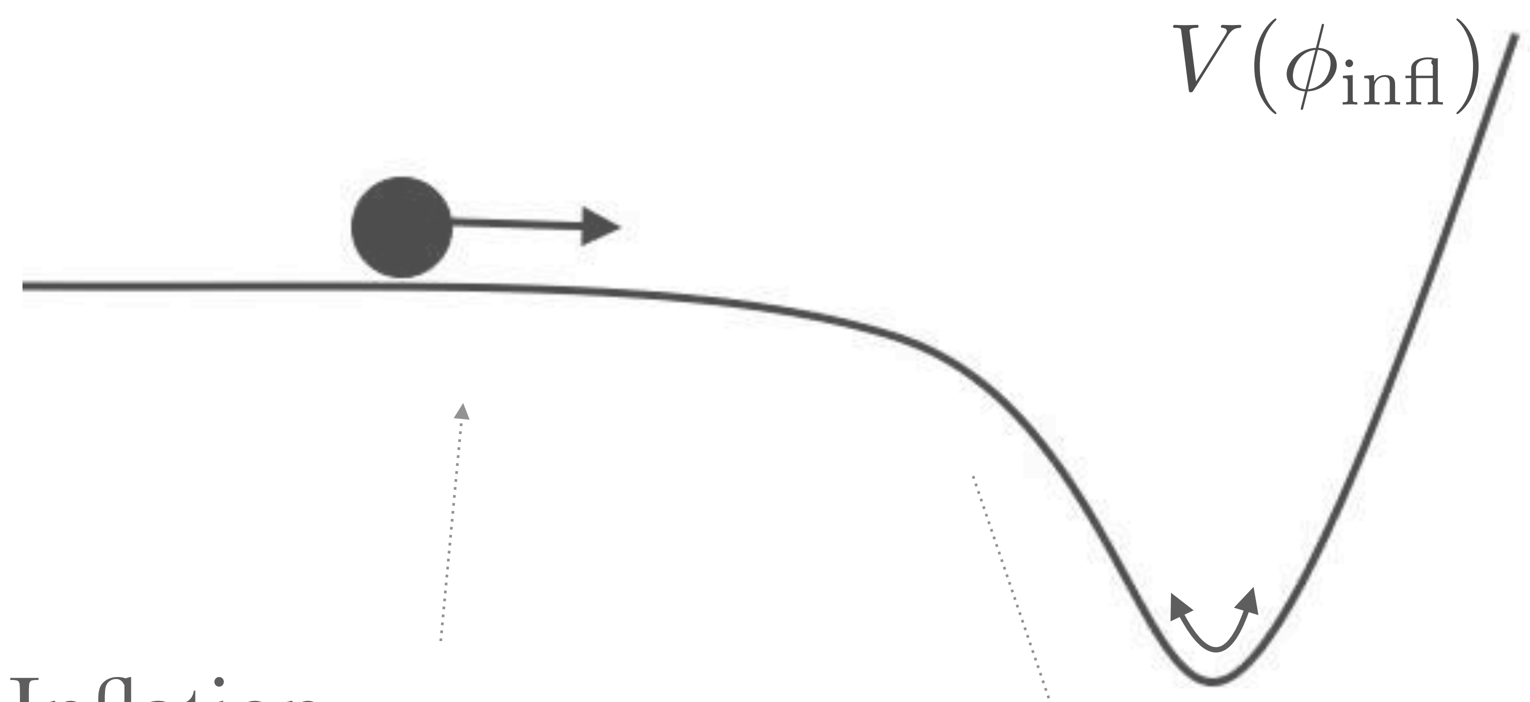
Review - lesson 3

Inflation (cont.)

Problems of the standard cosmological model

- Horizon problem
- Problem of the origin of structures
- Flatness problem
- Problem of the magnetic monopoles
- Initial singularity
- DM and DE

Inflation and graceful exit



Inflation

Causes the accelerated expansion - quasi-exponential

End of inflation

$$\epsilon \ll 1$$

$$|\delta| \ll 1$$

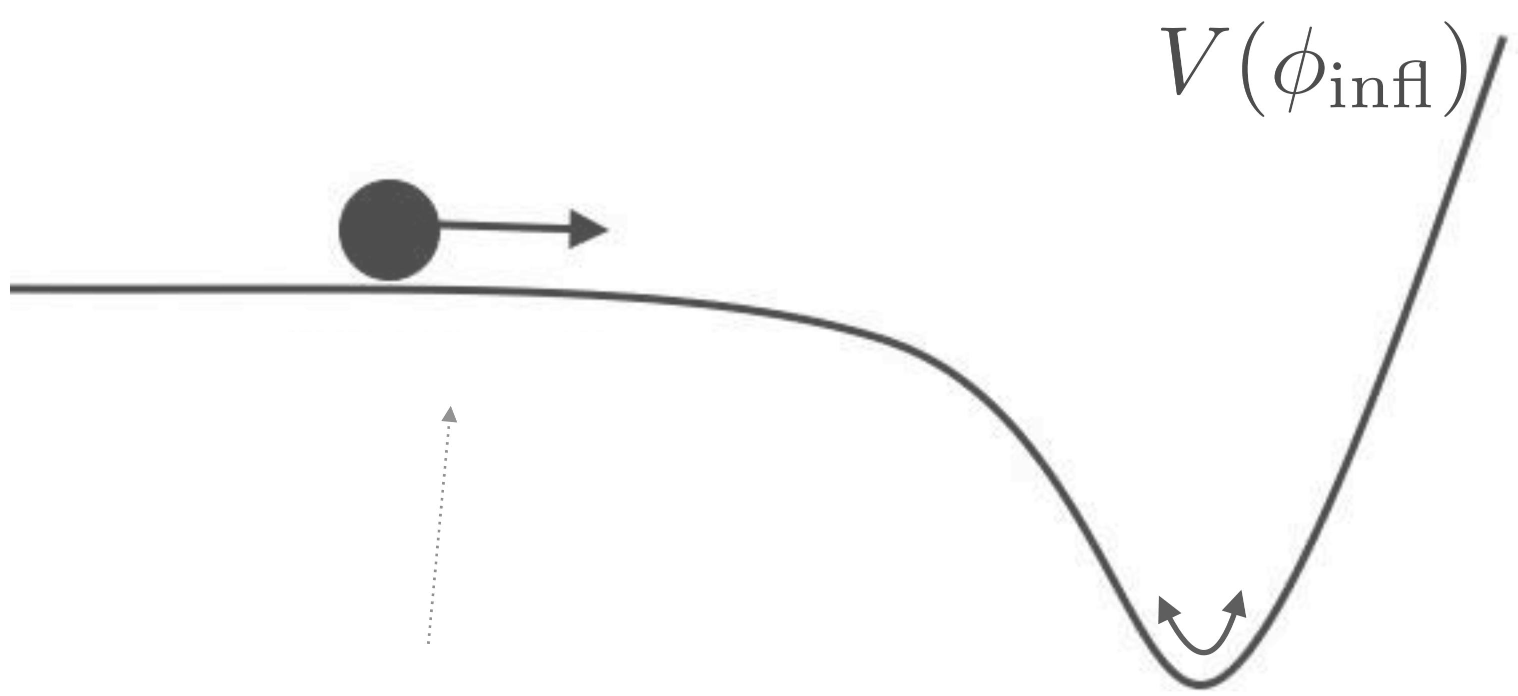
$$\epsilon \sim 1$$

Potential slow-roll parameters

$$\epsilon_{\rm v} \equiv \frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2 \,, \qquad |\eta_{\rm v}| \equiv M_{\rm pl}^2 \frac{|V''|}{V}$$

The inflationary mechanism has to have a period of quasi-accelerated expansion, BUT this has to end, to enter the radiation domination period $\rightarrow$ ***graceful exit***

After inflation - (p)reheating



Causes the accelerated expansion - quasi-exponential

PREHEATING

Non-perturbative

$$\ddot{\chi}_k + \left(k^2 + g^2\sigma^2 + 2g^2\sigma\Phi \,\sin mt\right)\chi_k = 0$$

Parametric resonance!!

REHEATING

Perturbative

To avoid that the universe ends up empty, the inflaton has to couple to Standard Model field

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi\rho_\phi$$

THERMALIZATION

Needs to lead into the SCM universe - in thermal equilibrium

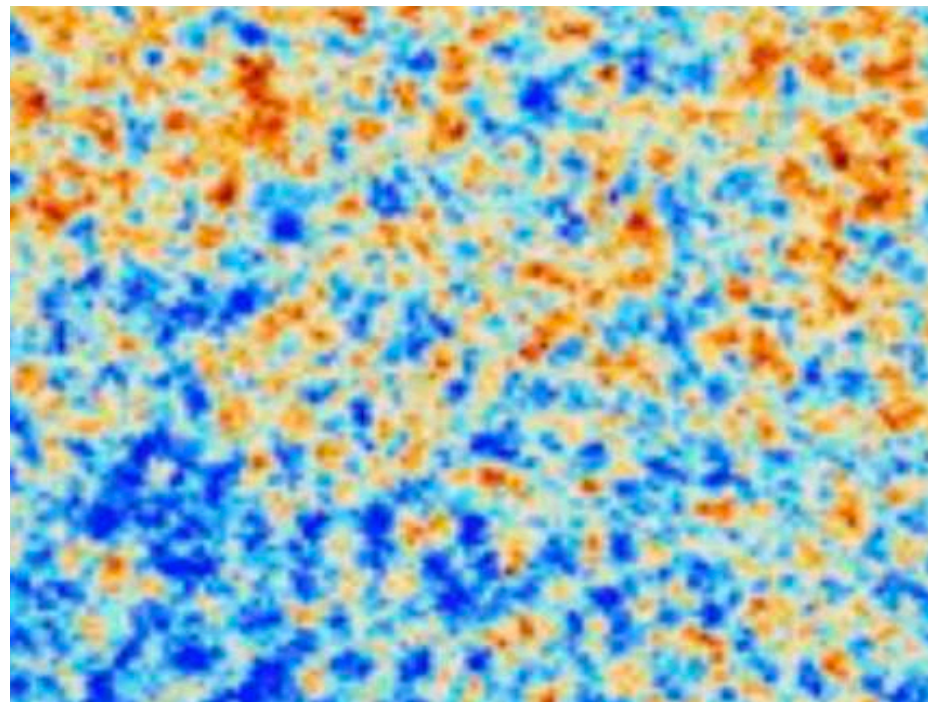
HOW?

Theory of cosmological perturbations

So far, we have treated the universe as perfectly homogeneous. To understand the formationand evolution of large-scale structures, we have to introduce inhomogeneities. As long as theseperturbations remain relatively small, we can treat them in perturbation theory

Fluido homogêneo e isotrópico

(fundo cosmológico)



So far, we have treated the universe as perfectly homogeneous.

$\bar{\rho}$

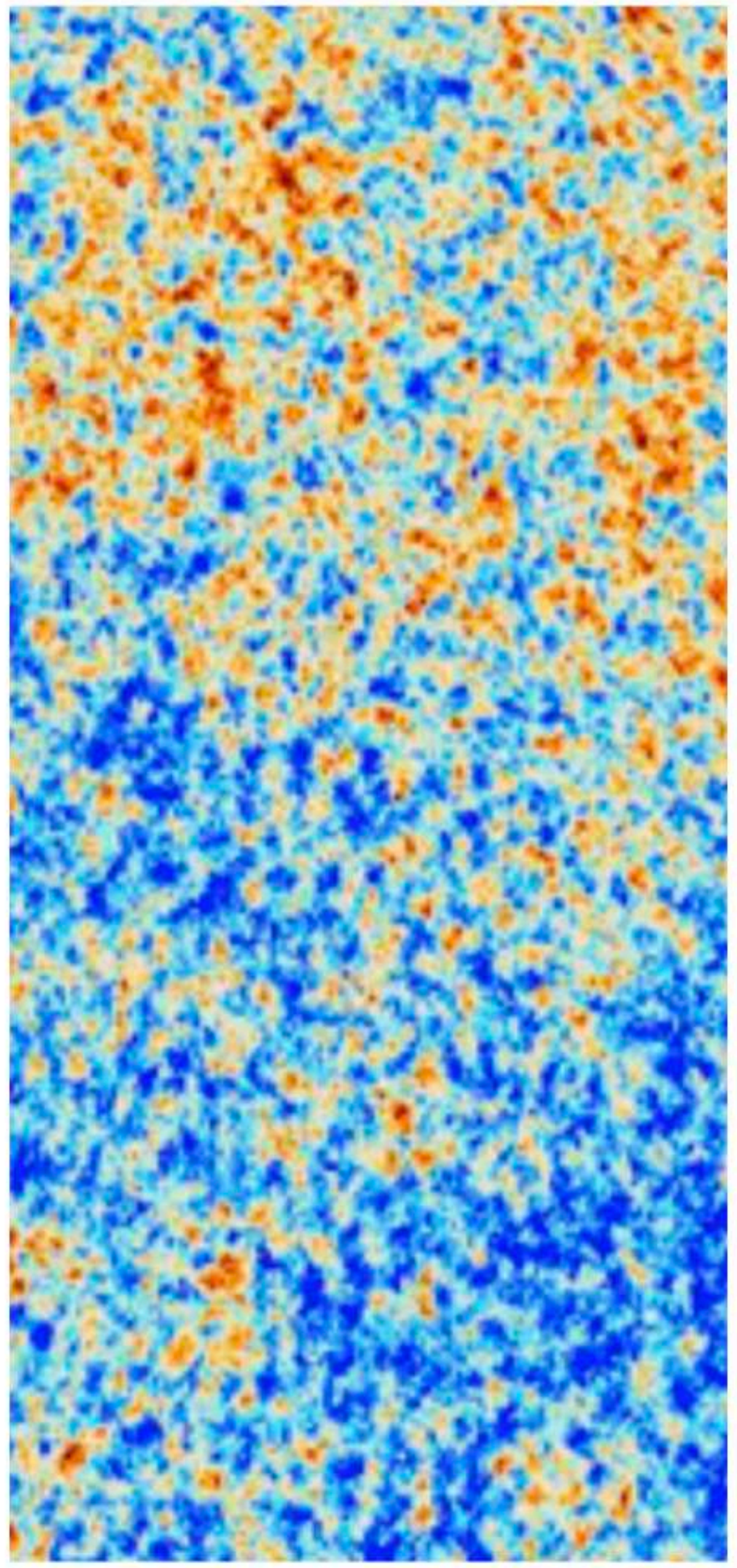
(Cosmological background)

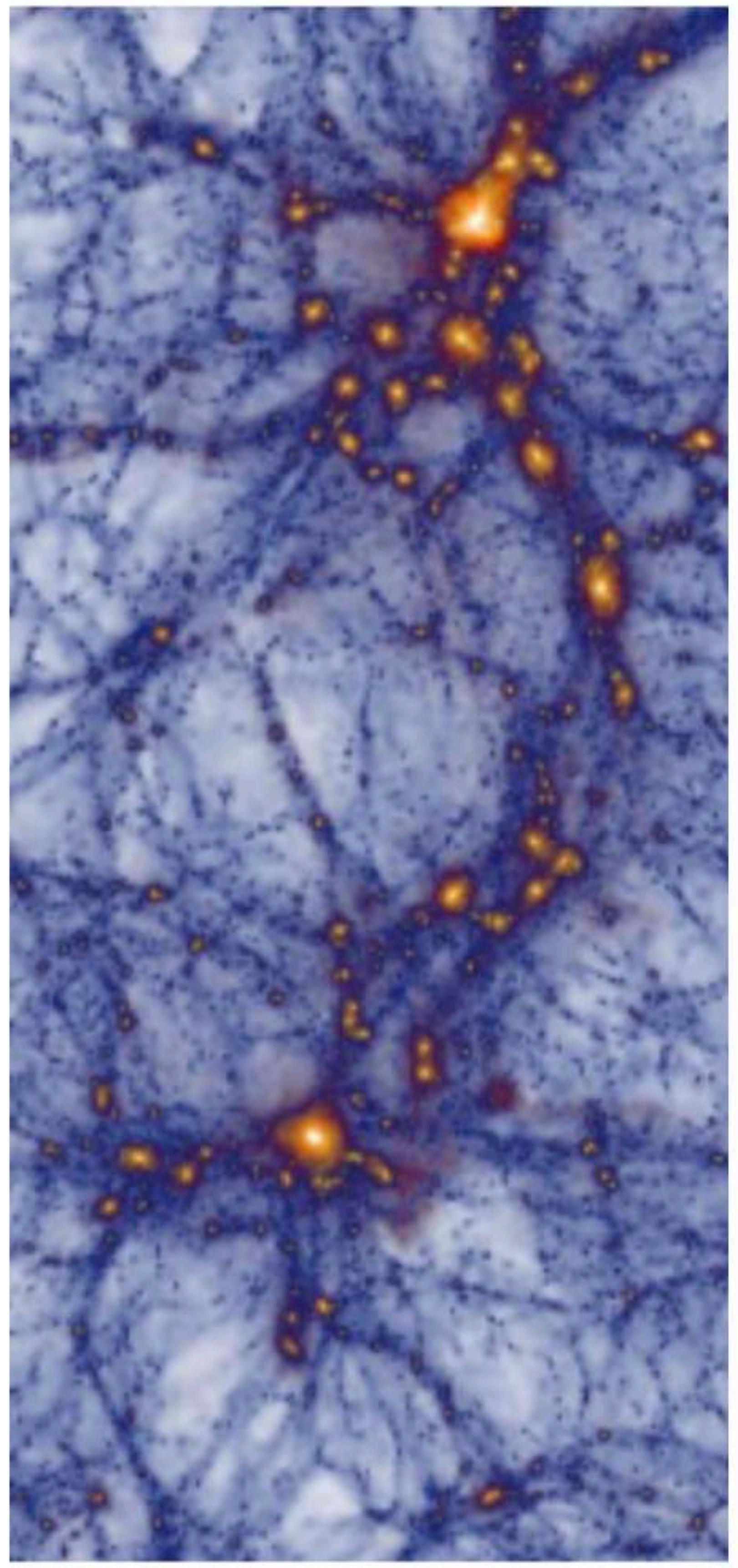
To understand the formation and evolution of large-scale structures, we have to introduce inhomogeneities.

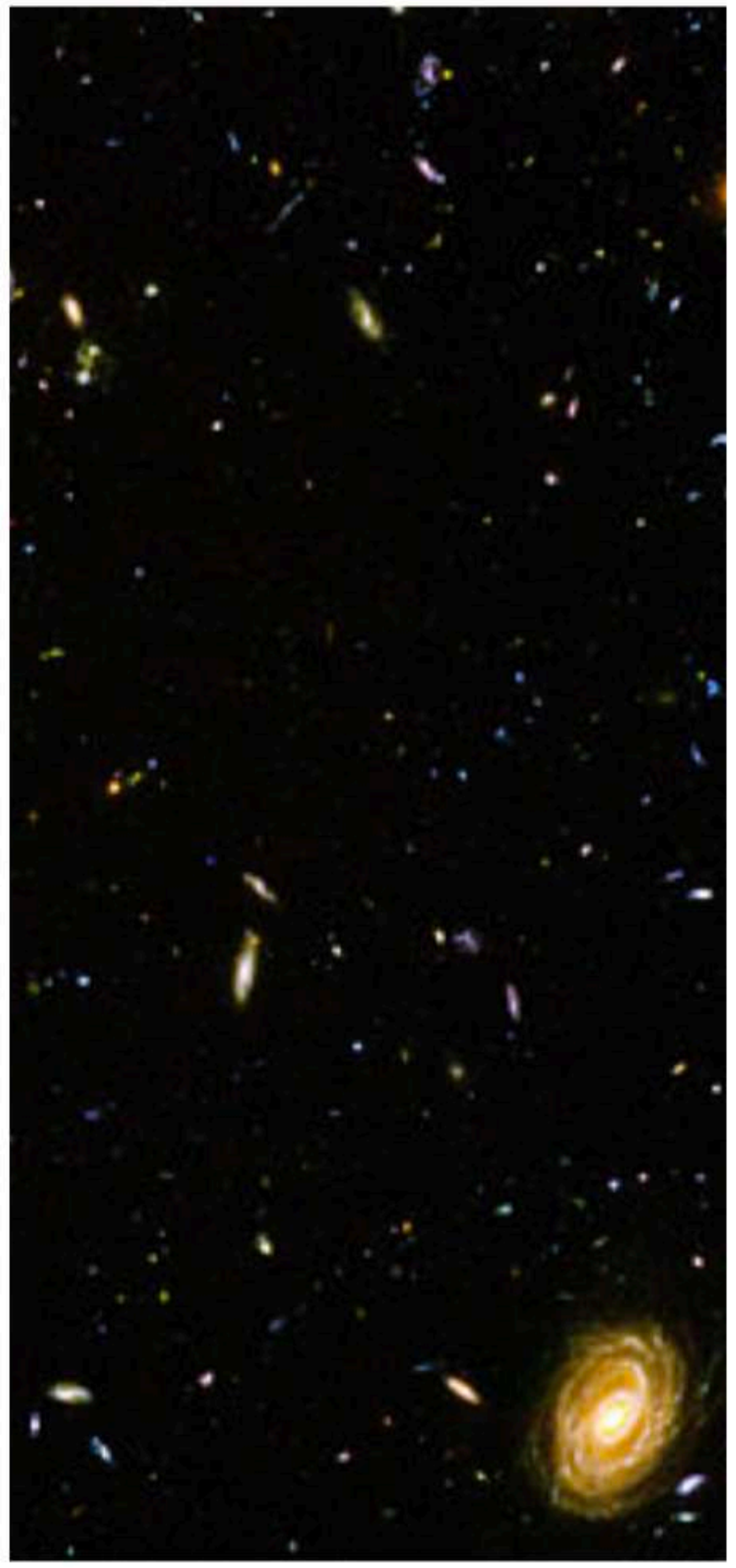
As long as these perturbations remain relatively small, we can treat them in perturbation theory

Theory of cosmological perturbations

Origin of the small perturbations → Small perturbations $\delta\rho \sim 10^{-5}\,\bar{\rho}$ → Macroscopic structures





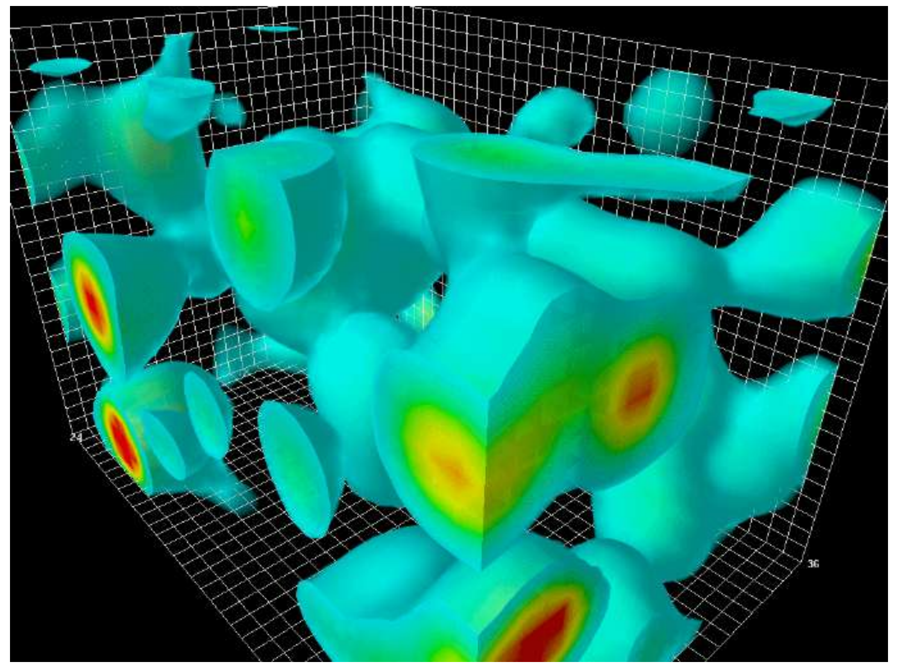


Let's study how these perturbations/inhomogeneities were formed and evolved (in the linear regime)

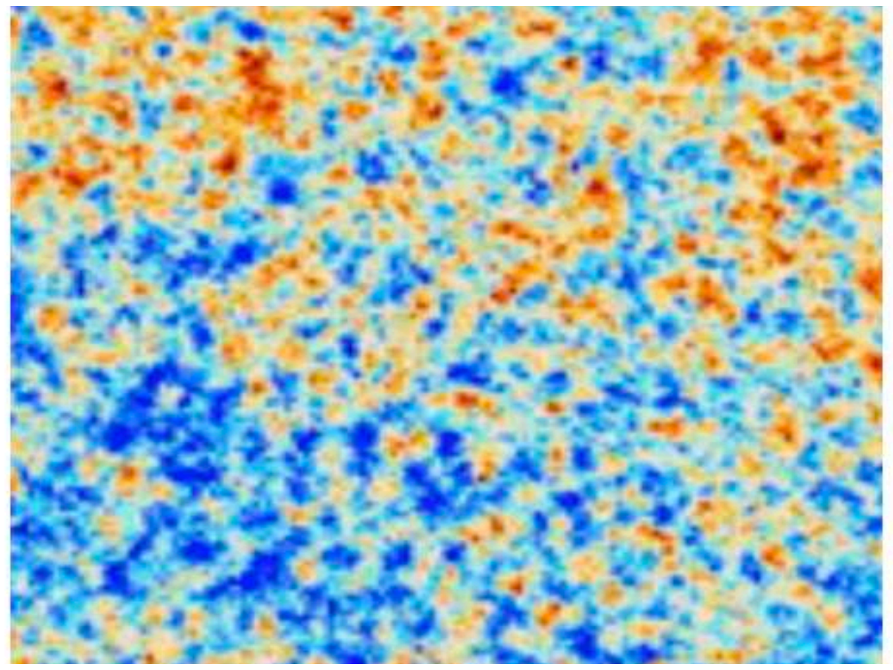
Crédito: D. Baumann

Theory of cosmological perturbations

Quantum theory of cosmological perturbations



Classical theory of cosmological perturbations

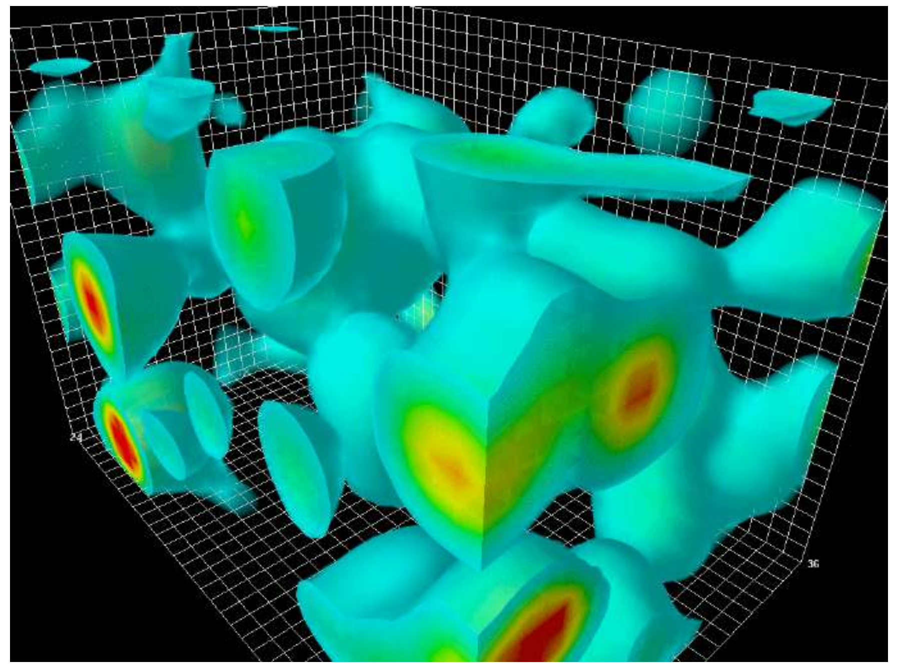


- Newtonian theory of cosm. perturbation
- Relativistic theory of cosm. perturbation

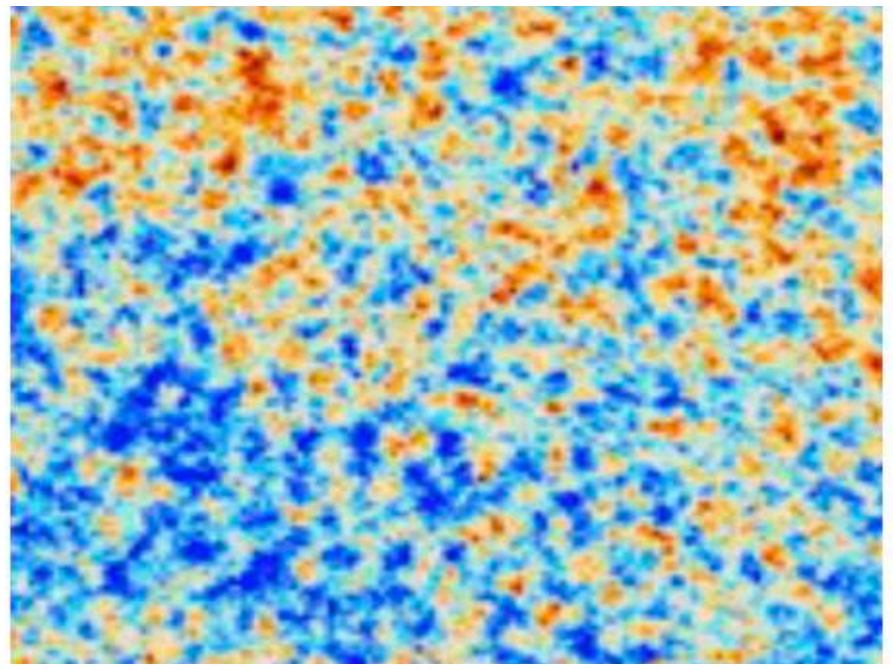
Crédito: D. Baumann

Theory of cosmological perturbations

Quantum theory of cosmological perturbations



Classical theory of cosmological perturbations



- Newtonian theory of cosm. perturbation
- Relativistic theory of cosm. perturbation

Crédito: D. Baumann

Newtonian theory of cosmological perturbations

Non-relativistic matter on scales not exceeding the Hubble horizon

On large scales matter can be described in a perfect fluid approximation

(continuity equation) $\frac{d\rho}{dt} + \nabla(\rho \mathbf{V}) = 0$

(Euller equation) $\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} + \frac{\nabla p}{\rho} + \nabla \Phi = 0$

(Poisson equation) $\triangle \Phi = 4\pi G \rho$

(conservation of entropy) $\frac{\partial S}{\partial t} + (\mathbf{V} \cdot \nabla) S = 0$

\+ EoS = complete set of equations to determine

$(\rho, \mathbf{V}, S, \Phi, p)$

In an expanding background: $\rho = \rho_0(t)\,,\ \mathbf{V} = \mathbf{V_0} = H(t) \cdot \mathbf{x}$

Newtonian theory of cosmological perturbations

Perturbing:

$\delta\rho(\mathbf{x},t) \ll \rho_0\,,\ldots$

[(NR) $p \ll \rho\,, (c_s, \delta V) \ll c$]

$$\rho = \rho_0(t) + \delta\rho(\mathbf{x},t)\,, \qquad \mathbf{V} = \mathbf{V}_0(t) + \delta\mathbf{V}(\mathbf{x},t)\,,$$
$$\Phi = \Phi_0(t) + \delta\Phi(\mathbf{x},t)\,, \qquad S = S_0(t) + \delta S(\mathbf{x},t)\,,$$

$$p(\mathbf{x},t) = p(\rho_0(t) + \delta\rho\,, S = S_0(t) + \delta S) = p_0 + \delta p(\mathbf{x},t)$$

In the linear approximation:

$$\delta p(\mathbf{x},t) = \left(\frac{\partial p}{\partial \rho}\right)_S \delta\rho + \left(\frac{\partial p}{\partial S}\right)_\rho \delta S = c_s^2 \delta\rho + \sigma \delta S$$

sound speed

Newtonian theory of cosmological perturbations

Perturbing:

$$\rho = \rho_0(t) + \delta\rho(\mathbf{x},t)\,, \qquad \mathbf{V} = \mathbf{V}_0(t) + \delta\mathbf{V}(\mathbf{x},t)\,,$$

$$\Phi = \Phi_0(t) + \delta\Phi(\mathbf{x},t)\,, \qquad S = S_0(t) + \delta S(\mathbf{x},t)\,,$$

$\delta\rho(\mathbf{x},t) \ll \rho_0\,, \ldots$

[(NR) $p \ll \rho\,, (c_s, \delta V) \ll c$]

$$p(\mathbf{x},t) = p(\rho_0(t) + \delta\rho\,, S = S_0(t) + \delta S) = p_0 + \delta p(\mathbf{x},t)$$

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sound speed

Linear hydrodynamical equations for the perturbations:

$$\frac{\partial \delta\rho(\mathbf{x},t)}{\partial t} + \rho_0 \nabla(\delta\mathbf{V}) = 0$$

$$\frac{\partial \delta\mathbf{V}(\mathbf{x},t)}{\partial t} + \frac{c_s^2}{\rho_0}\nabla\delta\rho + \frac{\sigma}{\rho_0}\nabla\delta S + \nabla\delta\Phi = 0$$

$$\frac{\partial \delta S(\mathbf{x},t)}{\partial t} = 0$$

$$\triangle\,\delta\Phi = 4\pi G\delta\rho$$

Newtonian theory of cosmological perturbations

Perturbing:

$\delta\rho(\mathbf{x},t) \ll \rho_0 \,, \ldots$

$[(\mathrm{NR})\, p \ll \rho \,, (c_s, \delta V) \ll c]$

$$\rho = \rho_0(t) + \delta\rho(\mathbf{x},t)\,, \qquad \mathbf{V} = \mathbf{V}_0(t) + \delta\mathbf{V}(\mathbf{x},t)\,,$$
$$\Phi = \Phi_0(t) + \delta\Phi(\mathbf{x},t)\,, \qquad S = S_0(t) + \delta S(\mathbf{x},t)\,,$$

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sound speed

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$$\frac{\partial \delta\mathbf{V}(\mathbf{x},t)}{\partial t} + \frac{c_s^2}{\rho_0}\nabla\delta\rho + \frac{\sigma}{\rho_0}\nabla\delta S + \nabla\delta\Phi = 0$$
$$\frac{\partial \delta S(\mathbf{x},t)}{\partial t} = 0 \longrightarrow \delta S(\mathbf{x},t) = \delta S(\mathbf{x})$$
$$\triangle\, \delta\Phi = 4\pi G \delta\rho$$

Newtonian theory of cosmological perturbations

Linear hydrodynamical equations for the perturbations:

$$\frac{\partial \delta\rho(\mathbf{x},t)}{\partial t} + \rho_0 \nabla(\delta \mathbf{V}) = 0$$

$$\frac{\partial \delta\mathbf{V}(\mathbf{x},t)}{\partial t} + \frac{c_s^2}{\rho_0}\nabla\delta\rho + \frac{\sigma}{\rho_0}\nabla\delta S + \nabla\delta\Phi = 0$$

$$\frac{\partial \delta S(\mathbf{x},t)}{\partial t} = 0 \quad \longrightarrow \quad \delta S(\mathbf{x},t) = \delta S(\mathbf{x})$$

$$\triangle\, \delta\Phi = 4\pi G \delta\rho$$

We can combine these equations :

$$\frac{\partial^2 \delta\rho(\mathbf{x},t)}{\partial t^2} - c_s^2 \triangle\, \delta\rho - 4\pi G \rho_0 \delta\rho = \sigma \triangle\, \delta S(\mathbf{x})$$

linear equation for $\delta\rho$ $\rightarrow$ δS acts as source

Newtonian theory of cosmological perturbations

$$\frac{\partial^2 \delta\rho(\mathbf{x},t)}{\partial t^2} - c_s^2 \triangle \delta\rho - 4\pi G\rho_0\delta\rho = \sigma \triangle \delta S(\mathbf{x})$$

Solutions:

- Adiabatic perturbations $\delta S = 0$

Coefficients do not depend on space. Fourier transform $\delta\rho(\mathbf{x},t) = \int \delta\rho_k(t)\exp(i\mathbf{kx})\frac{d^3k}{(2\pi)^3/2}$

$$\delta\ddot{\rho}_k + \underbrace{(k^2c_s^2 - 4\pi G\rho_0)}_{w_k^2}\delta\rho_k = 0 \qquad \Rightarrow \ \delta\rho \propto \exp(\pm i\omega_k t)$$

Jeans theory: when $\omega_k = 0 \longrightarrow \lambda_J = \frac{2\pi}{k_J} = c_s\left(\frac{\pi}{G\rho_0}\right)$

$$\begin{cases} \lambda < \lambda_J & \delta\rho \propto \sin(\omega_k t + \mathbf{kx} + \alpha) \\ \lambda > \lambda_J & \delta\rho \propto \exp(\pm|\omega|t) \end{cases}$$

(oscillatory)

(growth of inhomogeneities)

Newtonian theory of cosmological perturbations

$$\delta\ddot{\rho}_k + \underbrace{(k^2 c_s^2 - 4\pi G\rho_0)}_{w_k^2}\,\delta\rho_k = 0 \qquad \Rightarrow \;\; \delta\rho \propto \exp(\pm i\omega_k t)$$

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(oscillatory)

(growth of inhomogeneities)

$$k \to 0 \;\; \Rightarrow |\omega_k| t \to t/t_{gr}$$

characteristic collapse time for a region with initial density ρ_0 : $t_{gr} \equiv (4\pi G\; rho_0)^{-1/2}$

Newtonian theory of cosmological perturbations

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2} \triangle \delta - 4\pi G \rho_0 \delta = 0$$

Solutions: two adiabatic modes, two vector modes and one entropy mode

- Adiabatic perturbations $\delta S = 0$

- Vector perturbations $\delta = 0$ these vector perturbations describe shear motions of the media which do not disturb the energy density

- Entropy perturbations $\delta S \neq 0$ entropy perturbations can occur only in **multi-component fluids**.

For example, in a fluid consisting of baryons and radiation, the baryons can be distributed inhomogeneously on a homogeneous background of radiation. Entropy is equal to the number of photons per baryon and varies from place to place.

Newtonian theory of cosmological perturbations

Linear hydrodynamical equations for the perturbations:

$$\frac{\partial \delta\rho(\mathbf{x},t)}{\partial t} + \rho_0 \nabla(\delta \mathbf{V}) = 0$$
$$\frac{\partial \delta \mathbf{V}(\mathbf{x},t)}{\partial t} + \frac{c_s^2}{\rho_0}\nabla\delta\rho + \frac{\sigma}{\rho_0}\nabla\delta S + \nabla\delta\Phi = 0$$
$$\frac{\partial \delta S(\mathbf{x},t)}{\partial t} = 0$$
$$\triangle\,\delta\Phi = 4\pi G\delta\rho$$

$$\frac{\partial^2 \delta\rho(\mathbf{x},t)}{\partial t^2} - c_s^2 \triangle\,\delta\rho - 4\pi G\rho_0\delta\rho = \sigma \triangle\,\delta S(\mathbf{x})$$

In an expanding background:

$$\rho = \rho_0(t)\,,\ \ \mathbf{V} = \mathbf{V_0} = H(t)\cdot\mathbf{x}$$

ignoring entropy perturbations

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla\delta\mathbf{V} = 0$$
$$\frac{\partial \delta \mathbf{V}(\mathbf{x},t)}{\partial t} + H\,\delta\mathbf{V} + \frac{c_s^2}{a}\nabla\delta + \frac{1}{a}\nabla\delta\Phi = 0$$
$$\triangle\,\delta\Phi = 4\pi G a^2\rho_0\delta$$

$$\delta \equiv \delta\rho/\rho_0$$

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2}\triangle\,\delta - 4\pi G\rho_0\delta = 0$$

Newtonian theory of cosmological perturbations

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2} \triangle \delta - 4\pi G \rho_0 \delta = 0$$

Solutions:

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$$\ddot{\delta}_k + 2H\dot{\delta}_k + \underbrace{\left(\frac{c_s^2 k^2}{a^2} - 4\pi H \rho_0\right)}_{\omega_k^2}\delta_k = 0 \qquad \Rightarrow \quad \delta\rho \propto \exp(\pm i\omega_k t)$$

Jeans theory: when $\omega_k = 0 \longrightarrow \lambda_J^{\mathrm{phys}} = \frac{2\pi a}{k_J} = c_s\sqrt{\frac{\pi}{G\rho_0}}$

$$\begin{cases} \lambda \ll \lambda_J & \delta_k \propto \frac{1}{\sqrt{c_s a}}\exp\left(\pm k\int \frac{c_s dt}{a}\right) \\ \lambda \gg \lambda_J & \delta = C_1 H \int \frac{dt}{a^2 H^2} + C_2 H \end{cases}$$

(oscillatory)

(growth of inhomogeneities)

Newtonian theory of cosmological perturbations

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2} \triangle \delta - 4\pi G\rho_0 \delta = 0$$

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(oscillatory)

(growth of inhomogeneities)

$$\delta = C_1 t^{2/3} + C_2 t^{-1}$$

In an expanding universe, gravitational instability is much less efficient and the perturbation amplitude increases only as a power of time.

Newtonian theory of cosmological perturbations

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2} \triangle \delta - 4\pi G\rho_0\delta = 0$$

Solutions:

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(oscillatory)

(growth of inhomogeneities)

$$\delta = C_1 t^{2/3} + C_2 t^{-1}$$

If we want to obtain large inhomogeneities δ~1 today, we have to assume that at early times (for example, at redshifts z = 1000 - CMB) the inhomogeneities were already substantial (δ ~10^{-3} - we see $\delta \sim 10^{-5}$).

Theory of cosmological perturbations in GR

Perturbing: small perturbations around FRW

$$g_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \boldsymbol{x}) \qquad \text{(metric perturbation)}$$
$$T_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{T}_{\mu\nu}(\eta) + \delta T_{\mu\nu}(\eta, \boldsymbol{x}) \qquad \text{(matter perturbation)}$$

In GR, perturbing the matter component leads to/means perturbing the metric

Theory of cosmological perturbations in GR

Perturbing: small perturbations around FRW

$$g_{\mu\nu}(\eta,\boldsymbol{x}) = \bar{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta,\boldsymbol{x}) \qquad \text{(metric perturbation)}$$

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METRIC PERTURBATIONS

Since the perturbations are small we can linearize Einstein's equation. The perturbations of the metric lead to:

$$g_{\mu\nu} \to g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}\,; \qquad |\delta g_{\mu\nu}| \ll \left|g^{(0)}_{\mu\nu}\right|$$

$$\begin{aligned} ds^2 &= \left[g^{(0)}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta,\mathbf{x})\right] dx^\mu dx^\nu\,, \\ &= ds_0^2 + ds'^2\,, \end{aligned}$$

with

$$ds_0^2 = g^{(0)}_{\mu\nu}(\eta)\, dx^\mu dx^\nu = a^2(\eta)\left(d\eta^2 - \delta_{ij} dx^i dx^j\right) \longrightarrow \text{homogeneous and isotropic}$$

$$ds'^2 = \delta g_{\mu\nu}(\eta,\mathbf{x})\, dx^\mu dx^\nu$$

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

The perturbations of the metric lead to: $g_{\mu\nu} \rightarrow g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$; $|\delta g_{\mu\nu}| \ll \left|g^{(0)}_{\mu\nu}\right|$

$$ds^2 = \left[g^{(0)}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \mathbf{x})\right] dx^\mu dx^\nu \,, \\ = ds_0^2 + ds'^2 \,,$$

with

$$ds_0^2 = g^{(0)}_{\mu\nu}(\eta)\, dx^\mu dx^\nu = a^2(\eta)\left(d\eta^2 - \delta_{ij} dx^i dx^j\right)$$ → homogeneous and isotropic

$$ds'^2 = \delta g_{\mu\nu}(\eta, \mathbf{x})\, dx^\mu dx^\nu$$

Scalar, vector, tensor (SVT) decomposition

The most general perturbation around the FRW metric can be written as: $$\mathrm{d}s^2 = a^2(\eta)\left[(1+2\Psi)\mathrm{d}\eta^2 - 2B_i \mathrm{d}x^i \mathrm{d}\eta - (\delta_{ij} + h_{ij})\mathrm{d}x^i \mathrm{d}x^j\right]$$

It comes if I perturb every part of the metric

I want to know what is the irreducible number of variables needed to describe these perturbations

SYMMETRIES! FRW is inv. under rotation and translation. We can use this symmetry to classify the perturbations of the FRW metric

DOF Components of $g_{\mu\nu}$ - metric is symmetric $(g_{\mu\nu} = g_{\nu\mu})$ = perturbations will have 10 free DOF

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

Scalar, vector, tensor (SVT) decomposition

The most general perturbation around the FRW metric can be written as:

$$ds^2 = a^2(\eta)\Big[(1+2\Psi)d\eta^2 - 2B_i dx^i d\eta - (\delta_{ij} + h_{ij})dx^i dx^j\Big]$$

It comes if I perturb every part of the metric

Irreducible number of variables

$$B_i \equiv \underbrace{\partial_i B}_{scalar} + \underbrace{\hat{B}_i}_{vector:\ \partial^i \hat{B}_i = 0}$$

$$h_{ij} \equiv \underbrace{-2\Phi\,\delta_{ij}}_{scalar} + \underbrace{2\partial_{\langle i}\partial_{j\rangle} h}_{scalar} + \underbrace{2\partial_{(i}\hat{h}_{j)}}_{vector,\ \partial^i \hat{h}_i = 0} + \underbrace{2\hat{h}_{ij}}_{tensor,\ \partial^i \hat{h}_{ij} = \hat{h}^i{}_i = 0}$$

where $\partial_{\langle i}\partial_{j\rangle} h \equiv \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right) h$

$\partial_{(i}\hat{h}_{j)} \equiv \frac{1}{2}(\partial_i \hat{h}_j + \partial_j \hat{h}_i)$

$$\Rightarrow \quad \begin{aligned} \mathbf{10} \;=\;& \mathbf{4}\ \textit{scalar}\ \text{modes}: \ \Psi, B, \Phi, h \\ &+\mathbf{4}\ \textit{vector}\ \text{modes}: \ \hat{B}_i, \hat{h}_i \\ &+\mathbf{2}\ \textit{tensor}\ \text{modes}: \ \hat{h}_{ij} \end{aligned}$$

Theorem: At first order, scalars, vectors, and tensors don't mix

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

Scalar, vector, tensor (SVT) decomposition

With that, we can decompose our metric perturbations into scalar, vector and tensor perturbations

$$ds^2 = a^2(\eta)\left[(1+2\Psi)d\eta^2 - 2B_i dx^i d\eta - (\delta_{ij} + h_{ij})dx^i dx^j\right]$$

$$\delta g_{00}(\eta, \mathbf{x}) = 2a^2(\eta)\,\phi(\eta, \mathbf{x}) \longrightarrow \text{Scalar}$$

$$\delta g_{0i}(\eta, \mathbf{x}) = 2a^2(\eta)\,(B_{,i} + S_i) \longrightarrow \text{Vector}$$

$$\delta g_{ij}(\eta, \mathbf{x}) = 2a^2(\eta)\,(2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij})(\eta, \mathbf{x}) \longrightarrow \text{Tensor}$$

* Convention: Latin indices are raised and lowered with δ_{ij}: $h^i_i = \delta^{ij} h_{ij}$

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

Scalar, vector, tensor (SVT) decomposition

With that, we can decompose our metric perturbations into scalar, vector and tensor perturbations

$$\mathrm{d}s^2 = a^2(\eta)\Big[(1+2\Psi)\mathrm{d}\eta^2 - 2B_i\mathrm{d}x^i\mathrm{d}\eta - (\delta_{ij}+h_{ij})\mathrm{d}x^i\mathrm{d}x^j\Big]$$

$$\delta g_{00}(\eta,\mathbf{x}) = 2a^2(\eta)\,\phi(\eta,\mathbf{x}) \longrightarrow \text{Scalar}$$

$$\delta g_{0i}(\eta,\mathbf{x}) = 2a^2(\eta)\,(B_{,i}+S_i) \xrightarrow{S_i^{;i}=0} \text{Vector}$$

$$\delta g_{ij}(\eta,\mathbf{x}) = 2a^2(\eta)\,(2\psi\delta_{ij}+2E_{,ij}+F_{i,j}+F_{j,i}+h_{ij})(\eta,\mathbf{x}) \xrightarrow{h_i^i=0,\ h^i_{j,i}=0} \text{Tensor}$$

scalar vector tensor

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

Scalar, vector, tensor (SVT) decomposition

With that, we can decompose our metric perturbations into scalar, vector and tensor perturbations

$$ds^2 = a^2(\eta)\Big[(1+2\Psi)d\eta^2 - 2B_i dx^i d\eta - (\delta_{ij} + h_{ij})dx^i dx^j\Big]$$

How to construct:

- Vector

Scalar R $\xrightarrow{\text{Take the derivative}}$ $\dfrac{\partial R}{\partial x^i} = R_{,i}$ $\Rightarrow$ vector!

OR

Vector S_i with $S_i^{;i} = 0$

So the decomposition is irreducible

(Otherwise can decompose into a vectorial part with divergent =0 and a scalar

Theory of cosmological perturbations in GR

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OR

Vector S_i with $S_{i}^{;i} = 0$

$$B_i = \underbrace{\partial_i B}_{\text{scalar}} + \underbrace{\hat{B}_i}_{\text{vector}}, \quad \partial^i \hat{B}_i = 0.$$

So the decomposition is irreducible

(Otherwise can decompose into a vectorial part with divergent =0 and a scalar

Theory of cosmological perturbations in GR

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How to construct:

- Tensor

Scalar R: Take the derivative 2x → $\frac{\partial^2 R}{\partial x^i \partial x^j} = R_{,ij}$; Multiply by a tensor → $R\,\delta_{ij}$ $\Rightarrow$ Tensors!

OR

Vector S_i: Take the derivative → $\frac{\partial^2 S_i}{\partial x^i} = S_{i,j}$ with $S_i^{;i} = 0$

OR Tensor h_{ij} with $h^i_i = 0,\ h^i_{j,i} = 0$

So the decomposition is irreducible

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

Scalar, vector, tensor (SVT) decomposition

With that, we can decompose our metric perturbations into scalar, vector and tensor perturbations

Rank-2 symmetric tensor

So the decomposition is irreducible

$$h_{ij} = \underbrace{2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E}_{\text{scalar}} + \underbrace{2\partial_{(i}\hat{E}_{j)}}_{\text{vector}} + \underbrace{2\hat{E}_{ij}}_{\text{tensor}}$$

$$\partial^i \hat{E}_{ij} = 0. \qquad \partial^i \hat{E}_i = 0$$

How to construct:

- Tensor

Scalar R

Take the derivative 2x → $\frac{\partial^2 R}{\partial x^i \partial x^j} = R_{,ij}$

Multiply by a tensor → $R\,\delta_{ij}$

$\Rightarrow$ Tensors!

OR

Vector S_i

Take the derivative → $\frac{\partial^2 S_i}{\partial x^i} = S_{i,j}$

with $S_i^{;i} = 0$

OR Tensor h_{ij}

with $h_i^i = 0,\ h^i_{j,i} = 0$

So the decomposition is irreducible

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

Scalar, vector, tensor (SVT) decomposition

With that, we can decompose our metric perturbations into scalar, vector and tensor perturbations

$$\mathrm{d}s^2 = a^2(\eta)\left[(1+2\Psi)\mathrm{d}\eta^2 - 2B_i \mathrm{d}x^i \mathrm{d}\eta - (\delta_{ij} + h_{ij})\mathrm{d}x^i \mathrm{d}x^j\right]$$

$$\delta g_{00}(\eta, \mathbf{x}) = 2a^2(\eta)\,\phi(\eta, \mathbf{x}) \longrightarrow \text{Scalar}$$

$$\delta g_{0i}(\eta, \mathbf{x}) = 2a^2(\eta)\,(B_{,i} + S_i) \xrightarrow{S_i^{;i} = 0} \text{Vector}$$

$$\delta g_{ij}(\eta, \mathbf{x}) = 2a^2(\eta)\,(2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij})(\eta, \mathbf{x}) \xrightarrow{h_i^i = 0,\ h_{j,i}^i = 0} \text{Tensor}$$

scalar vector tensor

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

Scalar, vector, tensor (SVT) decomposition

With that, we can decompose our metric perturbations into scalar, vector and tensor perturbations - grouping all the scalars, vectors and tensors

$$ds^2 = a^2(\eta)\Big[(1+2\Psi)d\eta^2 - 2B_i dx^i d\eta - (\delta_{ij} + h_{ij})dx^i dx^j\Big]$$

- Scalar

$$\delta g_{\mu\nu}^{escalar} = a^2 \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2\left(\psi\delta_{ij} - E_{ij}\right) \end{pmatrix}$$

$$ds^2 = a^2(\eta)\left\{(1+2\phi)\, d\eta^2 + 2B_{,i} d\eta dx^i + \left[(1-2\psi)\,\delta_{ij} - 2E_{,ij}\right] dx^i dx^j\right\}$$

- Vector

$$\delta g_{\mu\nu}^{vetorial} = a^2 \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,j} + F_{j,i} \end{pmatrix}$$

$$ds^2 = a^2(\eta)\left[d\eta^2 + 2S_i d\eta dx^i - \left(\delta_{ij} - F_{i,j} - F_{j,i}\right) dx^i dx^j\right]$$

- Tensor

$$\delta g_{\mu\nu}^{tensorial} = a^2 \begin{pmatrix} 0 & 0 \\ 0 & -h_{ij} \end{pmatrix}$$

$$ds^2 = a^2(\eta)\left[d\eta^2 - \left(\delta_{ij} - h_{ij}\right) dx^i dx^j\right]$$

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

Therefore, we can decompose the perturbations of the metric into scalar, vector and tensor

$$ds^2 = a^2(\eta)\left[(1+2\Psi)d\eta^2 - 2B_i dx^i d\eta - (\delta_{ij} + h_{ij})dx^i dx^j\right]$$

- Scalar

$$\delta g_{\mu\nu}^{escalar} = a^2 \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2\left(\psi\delta_{ij} - E_{ij}\right) \end{pmatrix}$$

- Vector

$$\delta g_{\mu\nu}^{vetorial} = a^2 \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,j} + F_{j,i} \end{pmatrix}$$

- Tensor

$$\delta g_{\mu\nu}^{tensorial} = a^2 \begin{pmatrix} 0 & 0 \\ 0 & -h_{ij} \end{pmatrix}$$

And if we do that on top of a FRW universe:

$$\begin{aligned} \mathbf{10} \;=\; & \;\mathbf{4}\; \textit{scalar}\text{ modes}: \;\Psi, B, \Phi, h \\ & +\mathbf{4}\; \textit{vector}\text{ modes}: \;\hat{B}_i, \hat{h}_i \\ & +\mathbf{2}\; \textit{tensor}\text{ modes}: \;\hat{h}_{ij} \end{aligned}$$

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

Therefore, we can decompose the perturbations of the metric into scalar, vector and tensor

$$ds^2 = a^2(\eta)\left[(1+2\Psi)d\eta^2 - 2B_i dx^i d\eta - (\delta_{ij} + h_{ij})dx^i dx^j\right]$$

- Scalar

$$\delta g_{\mu\nu}^{escalar} = a^2 \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2\left(\psi\delta_{ij} - E_{ij}\right) \end{pmatrix}$$

- Vector

$$\delta g_{\mu\nu}^{vetorial} = a^2 \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,j} + F_{j,i} \end{pmatrix}$$

- Tensor

$$\delta g_{\mu\nu}^{tensorial} = a^2 \begin{pmatrix} 0 & 0 \\ 0 & -h_{ij} \end{pmatrix}$$

And if we do that on top of a FRW universe:

$$\begin{aligned} \mathbf{10} \;=\; & \mathbf{4}\ \textit{scalar}\ \text{modes}: \ \Psi, B, \Phi, h \\ & +\mathbf{4}\ \textit{vector}\ \text{modes}: \ \hat{B}_i, \hat{h}_i \\ & +\mathbf{2}\ \textit{tensor}\ \text{modes}: \ \hat{h}_{ij} \end{aligned}$$

BUT, not all of those DOF are physical! Gauge problem!

Theory of cosmological perturbations in GR

Gauge transformations

Since GR allows for a freedom in the choice of coordinate system, the metric changes for ≠ coordinates.

Consider the infinitesimal transformation:

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu \qquad \text{where} \qquad \xi^i = \xi^i_\perp + \varsigma^{,i}$$

$$\begin{aligned} \tilde{g}_{\mu\nu}\left(\tilde{x}^\rho\right) &= \frac{\partial x^\gamma}{\partial \tilde{x}^\mu}\frac{\partial x^\delta}{\partial \tilde{x}^\nu} g_{\gamma\delta}\left(\tilde{x}^\rho\right) \\ &\approx g^{(0)}_{\mu\nu}\left(x^\rho\right) + \delta g_{\mu\nu} - g^{(0)}_{\mu\delta}\xi^\delta_{,\nu} - g^{(0)}_{\gamma\nu}\xi^\gamma_{,\mu} \\ &= g^{(0)}_{\mu\nu}\left(\tilde{x}^\rho\right) + \delta\tilde{g}_{\mu\nu}\left(\tilde{x}^\rho\right) , \end{aligned}$$

This leads to:

$$\begin{aligned} \delta\tilde{g}_{00} &= \delta g_{00} - 2a\left(a\xi^0\right)' , \\ \delta\tilde{g}_{0i} &= \delta g_{0i} + a^2\left[\xi'_{\perp i} + \left(\varsigma' - \xi^0\right)_{,i}\right] , \\ \delta\tilde{g}_{ij} &= \delta g_{ij} + a^2\left[2\frac{a'}{a}\delta_{ij}\xi^0 + 2\varsigma_{,ij} + \left(\xi'_{\perp i,j} + \xi'_{\perp j,i}\right)\right] \end{aligned}$$

$\Rightarrow$

- Scalar: $\tilde{\phi} = \phi - \frac{1}{a}\left(a\xi^0\right)' , \quad \tilde{B} = B + \varsigma' - \xi^0 ,$
  $\tilde{\psi} = \psi + \frac{a'}{a}\xi^0 , \quad \tilde{E} = E + \varsigma .$
- Vector: $\tilde{S}_i = S_i + \xi'_{\perp i} , \quad \tilde{F}_i = F_i + \xi_{\perp i}$
- Tensor: invariant

Theory of cosmological perturbations in GR

Gauge transformations $\quad x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu \quad$ where $\quad \xi^i = \xi^i_\perp + \varsigma^{,i}$

This leads to:

$$\delta\tilde{g}_{00} = \delta g_{00} - 2a\left(a\xi^0\right)' ,$$
$$\delta\tilde{g}_{0i} = \delta g_{0i} + a^2\left[\xi'_{\perp i} + \left(\varsigma' - \xi^0\right)_{,i}\right] ,$$
$$\delta\tilde{g}_{ij} = \delta g_{ij} + a^2\left[2\frac{a'}{a}\delta_{ij}\xi^0 + 2\varsigma_{,ij} + \left(\xi'_{\perp i,j} + \xi'_{\perp j,i}\right)\right]$$

$$\Rightarrow$$

- Scalar: $\tilde{\phi} = \phi - \frac{1}{a}\left(a\xi^0\right)' , \quad \tilde{B} = B + \varsigma' - \xi^0 ,$
  $\tilde{\psi} = \psi + \frac{a'}{a}\xi^0 , \quad \tilde{E} = E + \varsigma .$
- Vector: $\tilde{S}_i = S_i + \xi'_{\perp i} , \quad \tilde{F}_i = F_i + \xi_{\perp i}$
- Tensor: invariant

This leads to: the metric perturbations **aren't uniquely defined**, but depend on our choice of coordinates or the "**gauge choice**".

Making a different choice of coordinates, can change the values of the perturbation variables.

➡ May introduce fictitious perturbations!

Theory of cosmological perturbations in GR

Gauge choice $\qquad x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu \qquad$ where $\qquad \xi^i = \xi^i_\perp + \varsigma^{,i}$

$$ds^2 = a^2(\eta)\left[d\eta^2 - 2\xi_i' d\tilde{x}^i d\eta - \left(\delta_{ij} + 2\partial_{(i}\xi_{j)}\right) d\tilde{x}^i d\tilde{x}^j\right]$$

Here, it looks like we have perturbation terms like: $\quad B_i = \xi_i' \,,\ \hat{E}_i = \xi_i \quad$ Like the ones we had before

BUT here they are just **gauge artifacts** (fictitious perturbations/not real)

IMPORTANT! Be careful!

Theory of cosmological perturbations in GR

Gauge choice $x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu$ where $\xi^i = \xi^i_\perp + \varsigma^{,i}$

How do we fix that?

1) work only with **gauge invariant** quantities

- Scalar $\Phi \equiv \phi - \frac{1}{a}\left[a\left(B - E'\right)\right]' \,, \quad \Psi \equiv \psi - \frac{a'}{a}\left(B - E'\right)$

- Vector $\bar{v}_i = S_i - F_i'$

Theory of cosmological perturbations in GR

Gauge choice $x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$ where $\xi^i = \xi^i_\perp + \varsigma^{,i}$

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- Vector $\bar{v}_i = S_i - F_i'$

2) choose/adopt a gauge

Theory of cosmological perturbations in GR

Gauge choice

2) choose/adopt a gauge

Newtonian gauge: $B = 0\,,\, E = 0\,,\; S_i = 0\,,$

$$ds^2 = a^2\,(\eta)\left[(1+2\phi)\,d\eta^2 - (1-2\psi - F_{i,j} - F_{j,i} - h_{ij})\,\delta_{ij}dx^i dx^j\right]$$

$$\Rightarrow \qquad \Phi = \phi\,, \quad \Psi = \psi\,, \quad \Phi_i = E_i^{'}$$

Conformal gauge: $\phi = 0\,, \quad B = 0$

Spatially-flat gauge:

Theory of cosmological perturbations in GR

Gauge choice

We will focus mostly on the scalar modes

$$\underset{\substack{\text{physical}\\ \text{scalar modes}}}{\mathbf{2}} = \mathbf{4} - \underset{\substack{\text{coordinate transformations:}\\ \eta \to \eta + T \text{ and } x^i \to x^i + \partial^i L}}{\mathbf{2}}$$

Next lesson

MATTER PERTURBATIONS

Theory of cosmological perturbations in GR (cont.)

$$g_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \boldsymbol{x}) \qquad \text{(metric perturbation)}$$

$$T_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{T}_{\mu\nu}(\eta) + \delta T_{\mu\nu}(\eta, \boldsymbol{x}) \qquad \text{(matter perturbation)}$$

Perturb the energy momentum tensor