Lesson 5:
Theory of cosmological perturbations (cont.)

Elisa G. M. Ferreira

Universidade de Sao Paulo & Kavli IPMU

Early universe cosmology, USP
01/Dec/2022

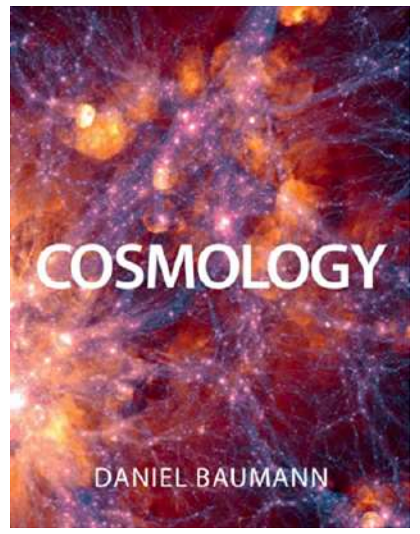
Image: Nasa,ESA, CSA, and STScI

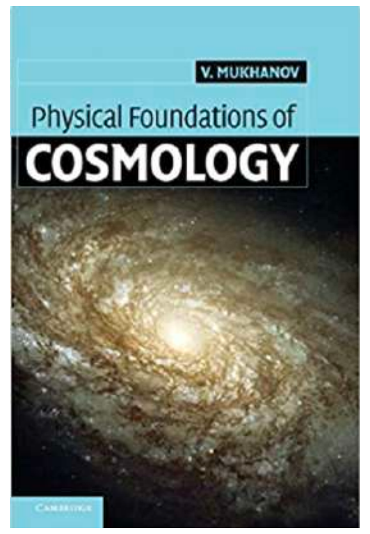
Early universe cosmology

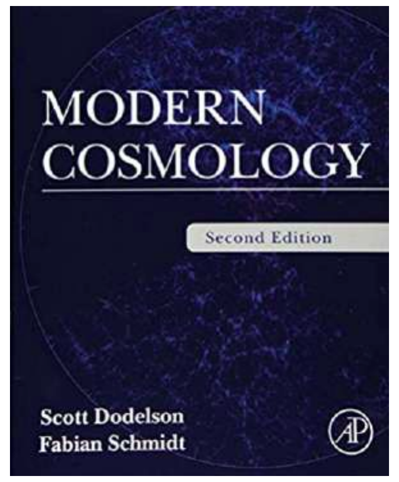
References:

- Daniel Baumann, Cosmology, Cambridge University Press, 2022.
- Viatcheslav Mukhanov, Physical Foundations of Cosmology, Cambridge University Press, 2005
- Review: "Theory of cosmological perturbations", V. F. Mukhanov, H. A. Feldman, R. H. Brandenberger, Physics Reports, 215, Issues 5–6 (1992), 203-333

- Daniel Baumann, TASI lectures on inflation
- (Recurso em português) Tese de mestrado Elisa G. M. Ferreira (capítulos 2 e 3)







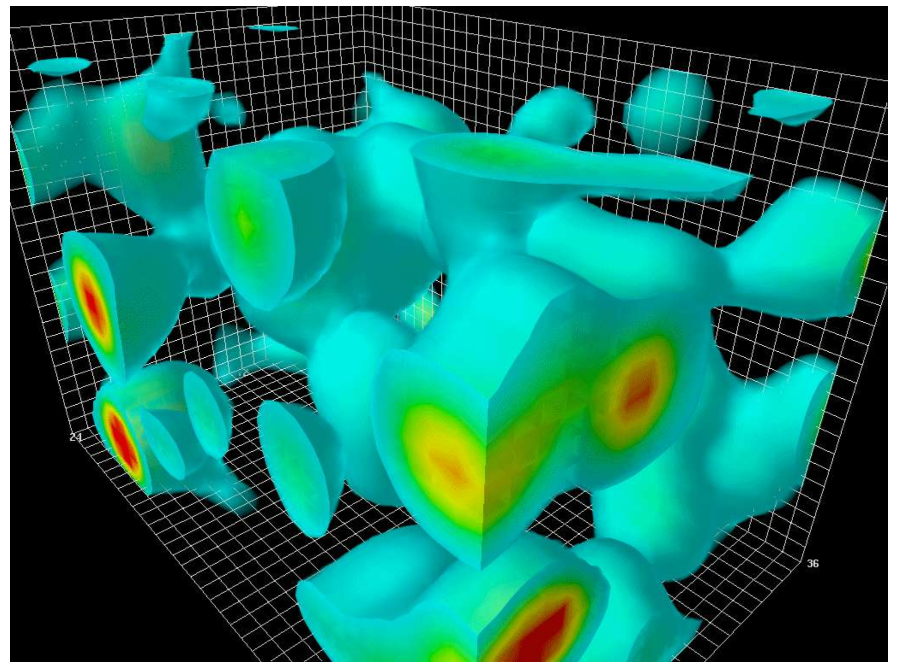
Questions: elisa@if.usp.br

Review - lesson 4

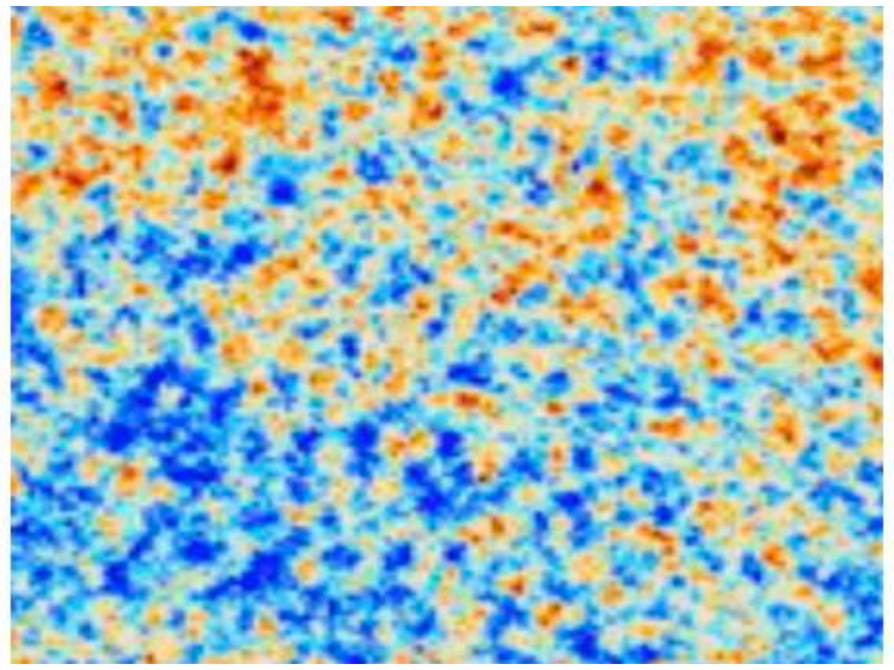
Theory of cosmological perturbations

Theory of cosmological perturbations

Quantum theory of cosmological perturbations



Classical theory of cosmological perturbations



- Newtonian theory of cosm. perturbation
- Relativistic theory of cosm. perturbation

Crédito: D. Baumann

Newtonian theory of cosmological perturbations

Perturbing:

$\delta\rho(\mathbf{x},t) \ll \rho_0\,,\ldots$

[(NR) $p \ll \rho\,, (c_s, \delta V) \ll c$]

$$\rho = \rho_0(t) + \delta\rho(\mathbf{x},t)\,, \qquad \mathbf{V} = \mathbf{V}_0(t) + \delta\mathbf{V}(\mathbf{x},t)\,,$$

$$\Phi = \Phi_0(t) + \delta\Phi(\mathbf{x},t)\,, \qquad S = S_0(t) + \delta S(\mathbf{x},t)\,,$$

$$p(\mathbf{x},t) = p(\rho_0(t) + \delta\rho\,, S = S_0(t) + \delta S) = p_0 + \delta p(\mathbf{x},t)$$

In the linear approximation:

$$\delta p(\mathbf{x},t) = \left(\frac{\partial p}{\partial \rho}\right)_S \delta\rho + \left(\frac{\partial p}{\partial S}\right)_\rho \delta S = c_s^2 \delta\rho + \sigma\delta S$$

sound speed

Newtonian theory of cosmological perturbations

Linear hydrodynamical equations for the perturbations:

$$\frac{\partial \delta\rho(\mathbf{x},t)}{\partial t} + \rho_0 \nabla(\delta \mathbf{V}) = 0$$
$$\frac{\partial \delta \mathbf{V}(\mathbf{x},t)}{\partial t} + \frac{c_s^2}{\rho_0}\nabla\delta\rho + \frac{\sigma}{\rho_0}\nabla\delta S + \nabla\delta\Phi = 0$$
$$\frac{\partial \delta S(\mathbf{x},t)}{\partial t} = 0$$
$$\triangle\,\delta\Phi = 4\pi G\delta\rho$$

$$\frac{\partial^2 \delta\rho(\mathbf{x},t)}{\partial t^2} - c_s^2 \triangle\,\delta\rho - 4\pi G\rho_0\delta\rho = \sigma\triangle\,\delta S(\mathbf{x})$$

In an expanding background:

$$\rho = \rho_0(t)\,,\ \ \mathbf{V} = \mathbf{V_0} = H(t)\cdot\mathbf{x}$$

ignoring entropy perturbations

$$\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla\delta\mathbf{V} = 0$$
$$\frac{\partial \delta \mathbf{V}(\mathbf{x},t)}{\partial t} + H\,\delta\mathbf{V} + \frac{c_s^2}{a}\nabla\delta + \frac{1}{a}\nabla\delta\Phi = 0$$
$$\triangle\,\delta\Phi = 4\pi G a^2\rho_0\delta$$

$$\delta \equiv \delta\rho/\rho_0$$

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2}\triangle\,\delta - 4\pi G\rho_0\delta = 0$$

Newtonian theory of cosmological perturbations

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2} \triangle \delta - 4\pi G \rho_0 \delta = 0$$

Solutions:

- Adiabatic perturbations $\delta S = 0$

Coefficients do not depend on space. Fourier transform $\delta\rho(\mathbf{x}, t) = \int \delta\rho_k(t) \exp(i\mathbf{kx}) \frac{d^3k}{(2\pi)^3/2}$

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \underbrace{\left(\frac{c_s^2 k^2}{a^2} - 4\pi H \rho_0\right)}_{\omega_k^2} \delta_k = 0 \qquad \Rightarrow \quad \delta\rho \propto \exp(\pm i\omega_k t)$$

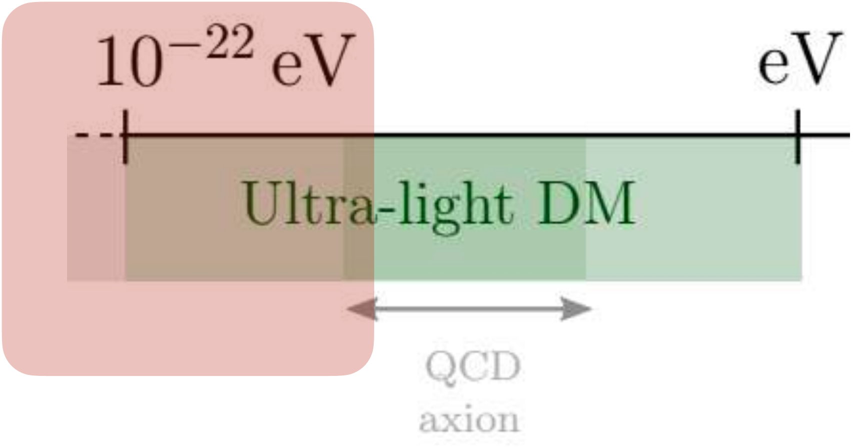
Jeans theory: when $\omega_k = 0 \longrightarrow \lambda_J^{\text{phys}} = \frac{2\pi a}{k_J} = c_s \sqrt{\frac{\pi}{G\rho_0}}$

$$\begin{cases} \lambda \ll \lambda_J & \delta_k \propto \frac{1}{\sqrt{c_s a}} \exp\left(\pm k \int \frac{c_s dt}{a}\right) \\ \lambda \gg \lambda_J & \delta = C_1 H \int \frac{dt}{a^2 H^2} + C_2 H \end{cases}$$

(oscillatory)

(growth of inhomogeneities)

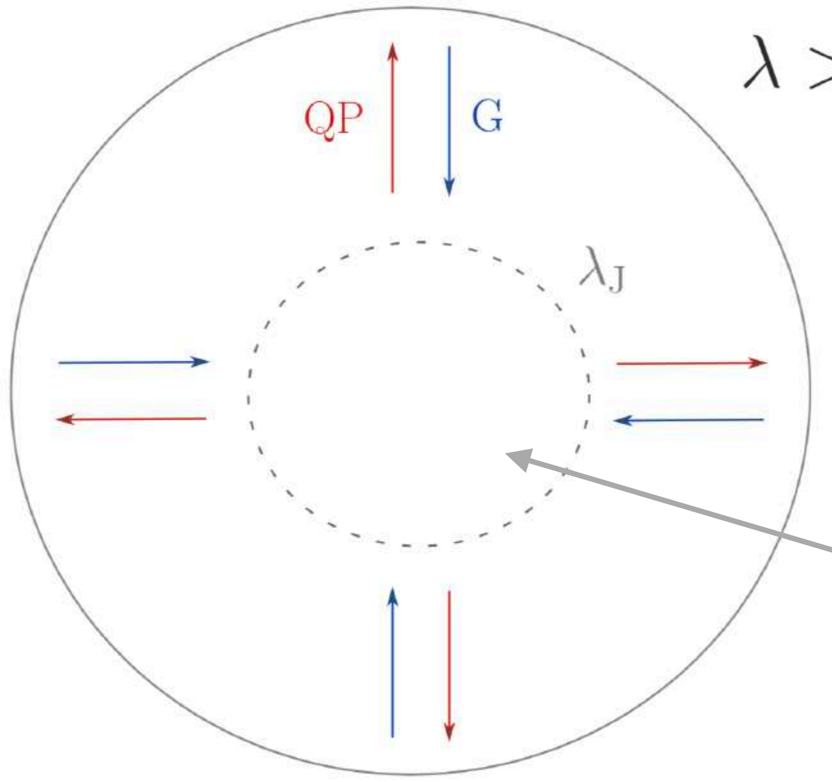
Structure formation - perturbation and stability



Finite clustering scale - no structure formation on small scales

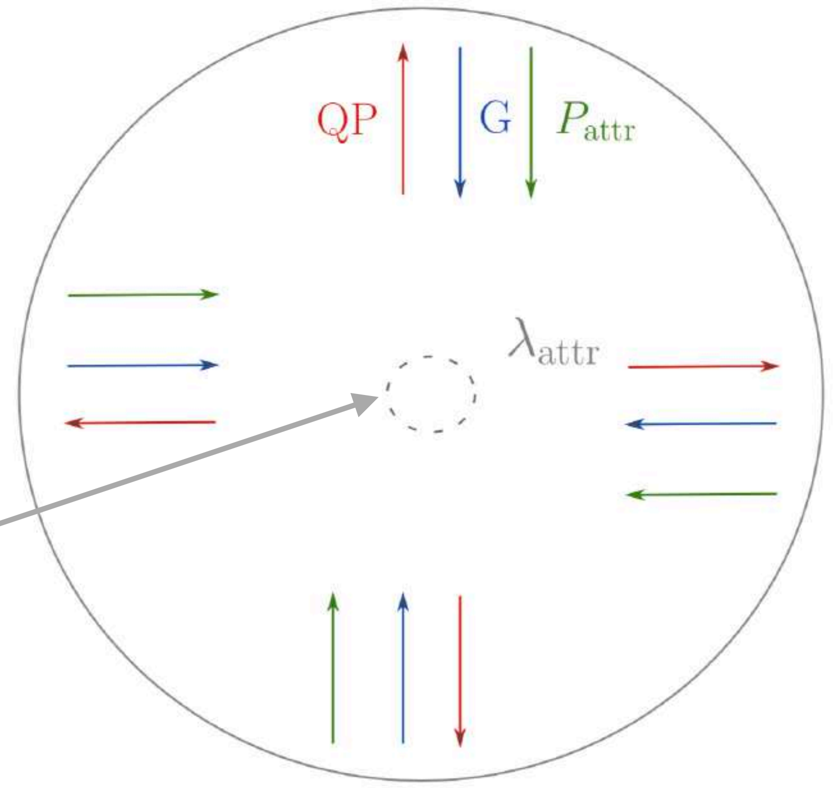
SIFDM

FDM

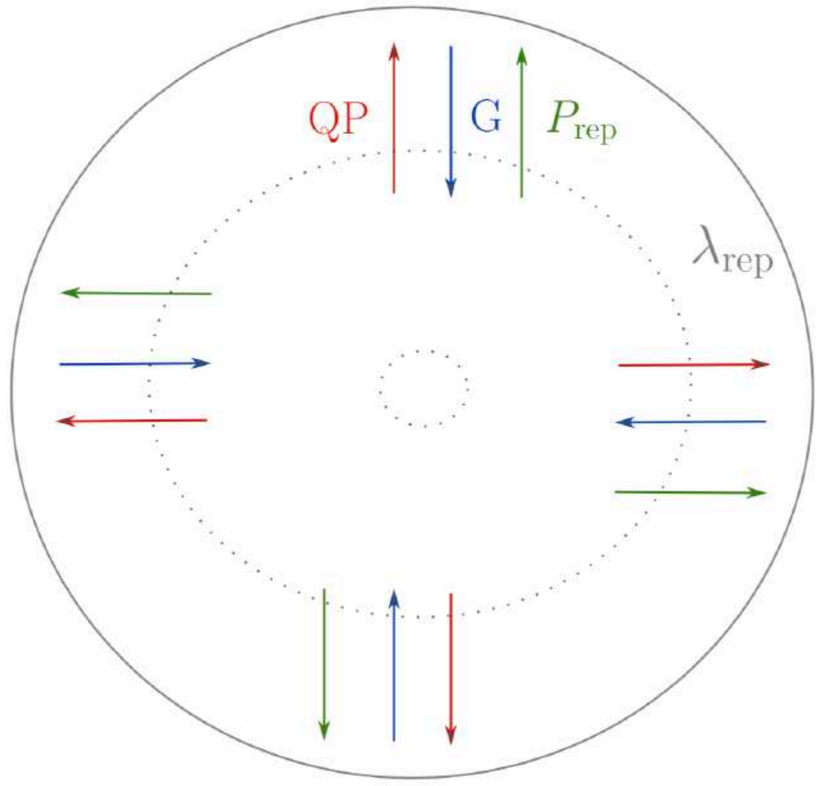


$\lambda > \lambda_J, \lambda_{attr}, \lambda_{rep} \longrightarrow$ CDM

ATTRACTIVE



REPULSIVE



QP wins -
NO structure formation
$\lambda < \lambda_J, \lambda_{attr}, \lambda_{rep}$

$g < 0$

$g > 0$

Finite size coherent core – Bose stars

$$\lambda_J = 55 \left(\frac{m}{10^{-22}\,\mathrm{eV}}\right)^{-1/2} \left(\frac{\rho}{\bar{\rho}}\right)^{-1/4} (\Omega_m h)^{-1/4}\,\mathrm{kpc}$$

$m \leq 10^{-20}\mathrm{eV} \;\Rightarrow\; \lambda_{dB} > \mathcal{O}(\mathrm{kpc})$ Galactic scales

For attractive interactions can only form localized clumps (solitons)

QCD axion: $m \sim 10^{-5}\,\mathrm{eV}$, $\lambda_a \sim -10^{-48}$ → $l_{soliton} \sim 10^{-5}\,\mathrm{kpc}$

Newtonian theory of cosmological perturbations

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2} \triangle \delta - 4\pi G\rho_0\delta = 0$$

Solutions:

- Adiabatic perturbations $\delta S = 0$

Coefficients do not depend on space. Fourier transform $\delta\rho(\mathbf{x}, t) = \int \delta\rho_k(t) \exp(i\mathbf{kx}) \frac{d^3k}{(2\pi)^3/2}$

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \underbrace{\left(\frac{c_s^2 k^2}{a^2} - 4\pi H\rho_0\right)}_{\omega_k^2} \delta_k = 0 \quad \Rightarrow \quad \delta\rho \propto \exp(\pm i\omega_k t)$$

Jeans theory: when $\omega_k = 0 \longrightarrow \lambda_J^{\text{phys}} = \frac{2\pi a}{k_J} = c_s\sqrt{\frac{\pi}{G\rho_0}}$

$$\begin{cases} \lambda \ll \lambda_J & \delta_k \propto \frac{1}{\sqrt{c_s a}} \exp\left(\pm k \int \frac{c_s dt}{a}\right) \\ \lambda \gg \lambda_J & \delta = C_1 H \int \frac{dt}{a^2 H^2} + C_2 H \end{cases}$$

(oscillatory)

(growth of inhomogeneities)

$$\delta = C_1 t^{2/3} + C_2 t^{-1}$$

In an expanding universe, gravitational instability is much less efficient and the perturbation amplitude increases only as a power of time.

Theory of cosmological perturbations in GR

Perturbing: small perturbations around FRW

$$g_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \boldsymbol{x}) \qquad \text{(metric perturbation)}$$

$$T_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{T}_{\mu\nu}(\eta) + \delta T_{\mu\nu}(\eta, \boldsymbol{x}) \qquad \text{(matter perturbation)}$$

In GR, perturbing the matter component leads to/means perturbing the metric

Theory of cosmological perturbations in GR

Perturbing: small perturbations around FRW

$$g_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \boldsymbol{x})$$ (metric perturbation)

$$T_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{T}_{\mu\nu}(\eta) + \delta T_{\mu\nu}(\eta, \boldsymbol{x})$$ (matter perturbation)

METRIC PERTURBATIONS

Since the perturbations are small we can linearize Einstein's equation. The perturbations of the metric lead to:

$$g_{\mu\nu} \to g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}\,;$$

$$|\delta g_{\mu\nu}| \ll \left|g^{(0)}_{\mu\nu}\right|$$

$$ds^2 = \left[g^{(0)}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \mathbf{x})\right] dx^\mu dx^\nu\,, \\ = ds_0^2 + ds'^2\,,$$

with

$$ds_0^2 = g^{(0)}_{\mu\nu}(\eta)\, dx^\mu dx^\nu = a^2(\eta)\left(d\eta^2 - \delta_{ij} dx^i dx^j\right)$$ ⟶ homogeneous and isotropic

$$ds'^2 = \delta g_{\mu\nu}(\eta, \mathbf{x})\, dx^\mu dx^\nu$$

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

The perturbations of the metric lead to: $g_{\mu\nu} \rightarrow g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}\,;$ $\quad |\delta g_{\mu\nu}| \ll \left|g^{(0)}_{\mu\nu}\right|$

$$\begin{aligned} ds^2 &= \left[g^{(0)}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \mathbf{x})\right] dx^\mu dx^\nu\,, \\ &= ds_0^2 + ds'^2\,, \end{aligned}$$

with

$ds_0^2 = g^{(0)}_{\mu\nu}(\eta)\, dx^\mu dx^\nu = a^2(\eta)\left(d\eta^2 - \delta_{ij} dx^i dx^j\right)$ ⟶ homogeneous and isotropic

$ds'^2 = \delta g_{\mu\nu}(\eta, \mathbf{x})\, dx^\mu dx^\nu$

Scalar, vector, tensor (SVT) decomposition

The most general perturbation around the FRW metric can be written as:

$$\mathrm{d}s^2 = a^2(\eta)\left[(1+2\Psi)\mathrm{d}\eta^2 - 2B_i \mathrm{d}x^i \mathrm{d}\eta - (\delta_{ij} + h_{ij})\mathrm{d}x^i \mathrm{d}x^j\right]$$

It comes if I perturb every part of the metric

DOF Components of $g_{\mu\nu}$ - metric is symmetric $(g_{\mu\nu} = g_{\nu\mu})$ = perturbations will have 10 free DOF

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

Scalar, vector, tensor (SVT) decomposition

With that, we can decompose our metric perturbations into scalar, vector and tensor perturbations - grouping all the scalars, vectors and tensors

$$ds^2 = a^2(\eta)\Big[(1+2\Psi)d\eta^2 - 2B_i dx^i d\eta - (\delta_{ij} + h_{ij})dx^i dx^j\Big]$$

- Scalar

$$\delta g_{\mu\nu}^{escalar} = a^2 \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2(\psi\delta_{ij} - E_{ij}) \end{pmatrix}$$

$$ds^2 = a^2(\eta)\left\{(1+2\phi)\, d\eta^2 + 2B_{,i} d\eta dx^i + \left[(1-2\psi)\,\delta_{ij} - 2E_{,ij}\right] dx^i dx^j\right\}$$

- Vector

$$\delta g_{\mu\nu}^{vetorial} = a^2 \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,j} + F_{j,i} \end{pmatrix}$$

$$ds^2 = a^2(\eta)\left[d\eta^2 + 2S_i d\eta dx^i - (\delta_{ij} - F_{i,j} - F_{j,i})\, dx^i dx^j\right]$$

- Tensor

$$\delta g_{\mu\nu}^{tensorial} = a^2 \begin{pmatrix} 0 & 0 \\ 0 & -h_{ij} \end{pmatrix}$$

$$ds^2 = a^2(\eta)\left[d\eta^2 - (\delta_{ij} - h_{ij})\, dx^i dx^j\right]$$

Theory of cosmological perturbations in GR

METRIC PERTURBATIONS

Therefore, we can decompose the perturbations of the metric into scalar, vector and tensor

$$ds^2 = a^2(\eta)\left[(1+2\Psi)\mathrm{d}\eta^2 - 2B_i \mathrm{d}x^i \mathrm{d}\eta - (\delta_{ij} + h_{ij})\mathrm{d}x^i \mathrm{d}x^j\right]$$

- Scalar

$$\delta g_{\mu\nu}^{escalar} = a^2 \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2\left(\psi\delta_{ij} - E_{ij}\right) \end{pmatrix}$$

- Vector

$$\delta g_{\mu\nu}^{vetorial} = a^2 \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,j} + F_{j,i} \end{pmatrix}$$

- Tensor

$$\delta g_{\mu\nu}^{tensorial} = a^2 \begin{pmatrix} 0 & 0 \\ 0 & -h_{ij} \end{pmatrix}$$

And if we do that on top of a FRW universe:

$$\begin{aligned} \mathbf{10} \ = \ & \ \mathbf{4}\ \textit{scalar}\ \text{modes}: \ \Psi, B, \Phi, h \\ & + \mathbf{4}\ \textit{vector}\ \text{modes}: \ \hat{B}_i, \hat{h}_i \\ & + \mathbf{2}\ \textit{tensor}\ \text{modes}: \ \hat{h}_{ij} \end{aligned}$$

BUT, not all of those DOF are physical! Gauge problem!

Theory of cosmological perturbations in GR

Gauge transformations $\quad x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu \quad$ where $\quad \xi^i = \xi^i_\perp + \varsigma^{,i}$

This leads to:

$$\delta\tilde{g}_{00} = \delta g_{00} - 2a\left(a\xi^0\right)' ,$$

$$\delta\tilde{g}_{0i} = \delta g_{0i} + a^2\left[\xi'_{\perp i} + \left(\varsigma' - \xi^0\right)_{,i}\right] ,$$

$$\delta\tilde{g}_{ij} = \delta g_{ij} + a^2\left[2\frac{a'}{a}\delta_{ij}\xi^0 + 2\varsigma_{,ij} + \left(\xi'_{\perp i,j} + \xi'_{\perp j,i}\right)\right]$$

$$\Rightarrow$$

- Scalar: $\tilde{\phi} = \phi - \frac{1}{a}\left(a\xi^0\right)' , \quad \tilde{B} = B + \varsigma' - \xi^0 ,$
  $\tilde{\psi} = \psi + \frac{a'}{a}\xi^0 , \quad \tilde{E} = E + \varsigma .$
- Vector: $\tilde{S}_i = S_i + \xi'_{\perp i} , \quad \tilde{F}_i = F_i + \xi_{\perp i}$
- Tensor: invariant

This leads to: the metric perturbations **aren't uniquely defined**, but depend on our choice of coordinates or the **"gauge choice"**.

Making a different choice of coordinates, can change the values of the perturbation variables.

→ May introduce fictitious perturbations!

Theory of cosmological perturbations in GR

Gauge choice $\qquad x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu \qquad$ where $\qquad \xi^i = \xi^i_\perp + \varsigma^{,i}$

$$ds^2 = a^2(\eta)\left[d\eta^2 - 2\xi_i' d\tilde{x}^i d\eta - \left(\delta_{ij} + 2\partial_{(i}\xi_{j)}\right) d\tilde{x}^i d\tilde{x}^j\right]$$

Here, it looks like we have perturbation terms like: $\quad B_i = \xi_i^{'}\,,\ \hat{E}_i = \xi_i \quad$ Like the ones we had before

BUT here they are just **gauge artifacts** (fictitious perturbations/not real)

IMPORTANT! Be careful!

Theory of cosmological perturbations in GR

Gauge choice $x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$ where $\xi^i = \xi^i_\perp + \varsigma^{,i}$

How do we fix that?

1) work only with **gauge invariant** quantities

- Scalar $\Phi \equiv \phi - \frac{1}{a}\left[a\left(B - E'\right)\right]'\,, \quad \Psi \equiv \psi - \frac{a'}{a}\left(B - E'\right)$

- Vector $\bar{v}_i = S_i - F_i'$

Theory of cosmological perturbations in GR

<u>Gauge choice</u> $x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu$ where $\xi^i = \xi^i_\perp + \varsigma^{,i}$

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- Scalar $\Phi \equiv \phi - \frac{1}{a}\left[a\left(B - E'\right)\right]' \,, \quad \Psi \equiv \psi - \frac{a'}{a}\left(B - E'\right)$

- Vector $\bar{v}_i = S_i - F_i'$

2) choose/adopt a gauge

Theory of cosmological perturbations in GR

Gauge choice

2) choose/adopt a gauge

Newtonian gauge: $B = 0\,,\, E = 0\,,\; S_i = 0\,,$

$$ds^2 = a^2\left(\eta\right)\left[\left(1+2\phi\right)d\eta^2 - \left(1-2\psi - F_{i,j} - F_{j,i} - h_{ij}\right)\delta_{ij}dx^i dx^j\right]$$

$$\Rightarrow \qquad \Phi = \phi\,, \quad \Psi = \psi\,, \quad \Phi_i = E_i^{'}$$

Conformal gauge: $\phi = 0\,, \quad B = 0$

Spatially-flat gauge:

Theory of cosmological perturbations in GR

Gauge choice

We will focus mostly on the scalar modes

$$\mathbf{2} \quad = \quad \mathbf{4} \quad - \quad \mathbf{2}$$

physical scalar modes

coordinate transformations: $\eta \to \eta + T$ and $x^i \to x^i + \partial^i L$

Theory of cosmological perturbations in GR

MATTER PERTURBATIONS

Theory of cosmological perturbations in GR (cont.)

$$g_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \boldsymbol{x}) \qquad \text{(metric perturbation)}$$

$$T_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{T}_{\mu\nu}(\eta) + \delta T_{\mu\nu}(\eta, \boldsymbol{x}) \qquad \text{(matter perturbation)}$$

Perturb the energy momentum tensor

Energy momentum tensor

GR cannot define the characteristics for the form of the energy momentum tensor

$$T_{\mu\nu} = \left(\begin{array}{c|c} T_{00} & T_{0i} \\ \hline T_{i0} & T_{ij} \end{array}\right) = \left(\begin{array}{c|c} \text{energy density} & \text{energy flux} \\ \hline \text{momentum density} & \text{stress tensor} \end{array}\right)$$

Given the previous hypotheses, the EMT has to obey the following properties:

A comoving observer only sees a homogeneous and isotropic universe if

- The scalar part is a function only of time. $T_{00} = \epsilon(t)$
- Vetor part is absent $T_{0i} = 0$
- Tensorial part is proportional to g_{ij} $T_{ij} = -P(t)\, g_{ij}$

$$T^{\mu}{}_{\nu} \equiv g^{\mu\lambda} T_{\lambda\nu} = \begin{pmatrix} \rho & & & \\ & -P & & \\ & & -P & \\ & & & -P \end{pmatrix}$$

Perfect fluid EMT

$$U_\mu = (1, 0, 0, 0)$$

For a ***general observer:***

$$T^{\mu}{}_{\nu} = (\rho + P) U^{\mu} U_{\nu} - P \delta^{\mu}_{\nu}$$

$\rho(t)$: densidade de energia

$P(t)$: pressão

U_μ : 4-velocidade relativa

In the fluid rest frame

Theory of cosmological perturbations in GR

MATTER PERTURBATIONS

Perturb the energy momentum tensor

$$T_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{T}_{\mu\nu}(\eta) + \delta T_{\mu\nu}(\eta, \boldsymbol{x})$$

$$T^{\mu}{}_{\nu} \equiv g^{\mu\lambda} T_{\lambda\nu} = \begin{pmatrix} \rho & & & \\ & -P & & \\ & & -P & \\ & & & -P \end{pmatrix}$$

For a fluid:

$$T^0{}_0 = \bar{\rho}(\eta) + \delta\rho \,, \qquad \delta \equiv \frac{\delta\rho}{\rho} \quad \text{(density contrast)}$$

$$T^i{}_0 = [\bar{\rho}(\eta) + \bar{P}(\eta)]\, v^i = q_i \qquad \text{(momentum density)}$$

$$T^i{}_j = -[\bar{P}(\eta) + \delta P]\, \delta^i_j - \Pi^i{}_j \,,$$

v_i is the bulk velocity and Π^i_j is a transverse and traceless tensor describing anisotropic stress.

Remember, this is for the sum of all component (baryons, photons, DM, …):

$$\delta\rho = \sum_a \delta\rho_a \,, \quad \delta P = \sum_a \delta P_a \,, \quad q^i = \sum_a q^i_{(a)} \,, \quad \Pi^{ij} = \sum_a \Pi^{ij}_{(a)}$$

(Velocities do not add)

Theory of cosmological perturbations in GR

MATTER PERTURBATIONS

$$T_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{T}_{\mu\nu}(\eta) + \delta T_{\mu\nu}(\eta, \boldsymbol{x})$$

Gauge transformations

Under the coordinate transformation $x^\mu \longrightarrow \tilde{x}^\mu = x^\mu + \xi^\mu$ where $\xi^i = \xi^i_\perp + \varsigma^{,i}$ the stress-energy tensor transform as

$$T^\mu{}_\nu(X) = \frac{\partial X^\mu}{\partial \tilde{X}^\alpha} \frac{\partial \tilde{X}^\beta}{\partial X^\nu} \tilde{T}^\alpha{}_\beta(\tilde{X})$$

For the components:

$$\delta\rho \mapsto \delta\rho - T\bar{\rho}' \ ,$$

$$\delta P \mapsto \delta P - T\bar{P}' \ ,$$

$$q_i \mapsto q_i + (\bar{\rho} + \bar{P})L_i' \ ,$$

$$v_i \mapsto v_i + L_i' \ ,$$

$$\Pi_{ij} \mapsto \Pi_{ij} \ .$$

Theory of cosmological perturbations in GR

MATTER PERTURBATIONS

$$T_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{T}_{\mu\nu}(\eta) + \delta T_{\mu\nu}(\eta, \boldsymbol{x})$$

Gauge transformations

Under the coordinate transformation $x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$ where $\xi^i = \xi^i_\perp + \varsigma^{,i}$ the stress-energy tensor transform as

$$T^\mu{}_\nu(X) = \frac{\partial X^\mu}{\partial \tilde{X}^\alpha} \frac{\partial \tilde{X}^\beta}{\partial X^\nu} \tilde{T}^\alpha{}_\beta(\tilde{X})$$

For the components:

$$\delta\rho \mapsto \delta\rho - T\bar{\rho}' ,$$

$$\delta P \mapsto \delta P - T\bar{P}' ,$$

$$q_i \mapsto q_i + (\bar{\rho} + \bar{P})L_i' ,$$

$$v_i \mapsto v_i + L_i' ,$$

$$\Pi_{ij} \mapsto \Pi_{ij} .$$

Gauge choice

1) work only with **gauge invariant** quantities $\bar{\rho}\Delta \equiv \delta\rho + \bar{\rho}'(v + B)$

2) choose/adopt a gauge

Uniform density gauge: We can use the freedom in the time-slicing to set the total density perturbation to zero $\delta\rho = 0$

Comoving gauge: scalar momentum density to vanish $q = 0$

Theory of cosmological perturbations in GR

MATTER PERTURBATIONS

$$T_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{T}_{\mu\nu}(\eta) + \delta T_{\mu\nu}(\eta, \boldsymbol{x})$$

If we model inflation with a scalar field, we can write the matter perturbations using only the matter fields:

$$\varphi(\eta, \mathbf{x}) \to \varphi(\eta) + \delta\varphi(\eta, \mathbf{x})$$

Under gauge transformations, it transforms as: $\delta\tilde{\varphi} = \delta\varphi + \varphi' \xi^0$

Gauge invariant variable: $\chi \equiv \delta\varphi + \left(B - E'\right)\varphi'$

$$\begin{aligned}
\delta T^0_0 &= -\varphi'\chi' - a^2 V_{,\varphi}\chi + \Phi\varphi' \,, \\
\delta T^0_i &= -\partial_i\left(\varphi'\chi\right) \,, \\
\delta T^i_j &= -\left(\varphi'^2\Phi + a^2 V_{,\varphi}\chi - \varphi'\chi'\right)\delta^i_j
\end{aligned}$$

Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

In the linear regime, EE are given by:

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}$$

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$: Tensor de Einstein

$R_{\mu\nu}$: Tensor de Ricci

$g_{\mu\nu}$: Métrica

R : Escalar de Ricci

$T_{\mu\nu}$: Tensor Energia-Momentum

Background, in a FRW universe:

$$G_0^{0(0)} = \frac{3\mathcal{H}^2}{a^2}\,, \quad G_i^{0(0)} = 0\,, \quad G_j^{i(0)} = \frac{1}{a^2}\left(2\mathcal{H}' + \mathcal{H}^2\right)\delta^i_j$$

$$\Rightarrow \qquad T_i^{0(0)} = 0\,, \quad T_j^{i(0)} \propto \delta^i_j$$

Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

In the linear regime, EE are given by:

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}$$

SVT:

Scalar modes:

$$\delta G^0_0 = \nabla^2\Psi - 3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) = 4\pi G a^2 \delta T^0_0\,,$$
$$\delta G^0_i = \left(\Psi' + \mathcal{H}\Phi\right)_{,i} = 4\pi G a^2 \delta T^0_i\,,$$
$$\delta G^i_j = \left[\Psi'' + \mathcal{H}\left(2+\Phi\right)' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + \frac{1}{2}\nabla^2\left(\Phi-\Psi\right)\right]\delta^i_j - \frac{1}{2}\left(\Phi-\Psi\right)^{;i}_{,j}$$
$$= -4\pi G a^2 \delta T^i_j\,.$$

Vector modes:

$$\delta G^0_i = 16\pi G a^2 \delta T^0_i\,,$$
$$\delta G^i_j = \left(\bar{v}_{i,j} + \bar{v}_{j,i}\right)' + 2\mathcal{H}\left(\bar{v}_{i,j} + \bar{v}_{j,i}\right) = -16\pi G a^2 \delta T^i_{j(V)}\,,$$

Tensor modes:

$$\delta G^i_j = \left(h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij}\right) = 16\pi G a^2 \delta T^i_{j(T)}$$

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$$= -4\pi G a^2 \delta T^i_j\,.$$

FRW universe:

$$\delta T^i_j \propto \delta^i_j$$

$$\boxed{\Phi = \Psi}$$

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Theory of cosmological perturbations in GR

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$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}$$

Scalar modes:

$$\nabla^2\Phi - 3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right) = 4\pi G\left(\varphi'\chi' + a^2 V_{,\varphi}\chi - \Phi\varphi'\right),$$
$$\Phi' + \mathcal{H}\Phi = 4\pi G\,\varphi'\chi,$$
$$\Phi'' + 3\mathcal{H}\Phi' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\Phi = 4\pi G\left(\varphi'\chi' - \varphi'^2\Phi - a^2 V_{,\varphi}\chi\right).$$

The ***scalar modes*** are the ONLY ones coupled to matter.

Relevant modes for structure formation!

Combine them (using KG equation):

$$\Phi'' + 2\left(\mathcal{H} - \frac{\varphi''}{\varphi'}\right)\Phi' - \nabla^2\Phi + 2\left(\mathcal{H}' - \mathcal{H}\frac{\varphi''}{\varphi'}\right)\Phi = 0,$$

Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

Scalar modes:

$$\Phi'' + 2\left(\mathcal{H} - \frac{\varphi''}{\varphi'}\right)\Phi' - \nabla^2\Phi + 2\left(\mathcal{H}' - \mathcal{H}\frac{\varphi''}{\varphi'}\right)\Phi = 0\,,$$

Change of variable $u \equiv \frac{a}{\varphi'}\Phi\,, \quad \theta \equiv \mathcal{H}/a\varphi'$

$$\rightarrow \quad u'' - \nabla^2 u - \underbrace{\left(\frac{\theta''}{\theta}\right)}_{m^2_{eff}} u = 0\,.$$

Fourier transform: $u\left(\mathbf{x},\eta\right) = \int \frac{d^3k}{(2\pi)^{3/2}} u_{\mathbf{k}}\left(\eta\right) \mathrm{e}^{i\mathbf{k}.\mathbf{x}}$

$\mathbf{x} = \mathrm{l}/a$

$\mathbf{k} = a\mathbf{k}_{fis}$

$$\rightarrow \quad \boxed{u''_{\mathbf{k}} + \left(k^2 - \frac{\theta''}{\theta}\right)u_{\mathbf{k}} = 0}$$

Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

$$u''_{\mathbf{k}} + \left(k^2 - \frac{\theta''}{\theta}\right) u_{\mathbf{k}} = 0$$

Scalar modes:

1st case: small wavelength $k^2 \gg \theta''/\theta$.

$$u_{\mathbf{k}}(\eta) = A^+_{\mathbf{k}} e^{ik\eta} + A^-_{\mathbf{k}} e^{-ik\eta}$$

$$\Phi_{\mathbf{k}}(t, \mathbf{x}) \simeq \dot{\varphi} e^{ik\mathbf{x}} \left[A^+_{\mathbf{k}} \mathrm{sen}\left(k \int \frac{dt}{a}\right) + A^-_{\mathbf{k}} \cos\left(k \int \frac{dt}{a}\right)\right]$$

Perturbations feel the expansion of the universe but not the space-time curvature.

Physical wavelength are **stretched** with the expansion of the universe

Theory of cosmological perturbations in GR

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Physical wavelength are **stretched** with the expansion of the universe

Perturbations can be stretched until they are larger than the Hubble radius if the grow faster than the curvature radius of the universe

$\Longrightarrow$ Inflation!

Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

Scalar modes:

$$u_{\mathbf{k}}'' + \left(k^2 - \frac{\theta''}{\theta}\right) u_{\mathbf{k}} = 0$$

1st case: small wavelength $k^2 \gg \theta''/\theta$

$$u_{\mathbf{k}}(\eta) = A_{\mathbf{k}}^{+} \mathrm{e}^{ik\eta} + A_{\mathbf{k}}^{-} \mathrm{e}^{-ik\eta}$$

$$\Phi_{\mathbf{k}}(t, \mathbf{x}) \simeq \dot{\varphi} \mathrm{e}^{ik\mathbf{x}} \left[A_{\mathbf{k}}^{+} \mathrm{sen}\left(k \int \frac{dt}{a}\right) + A_{\mathbf{k}}^{-} \cos\left(k \int \frac{dt}{a}\right)\right]$$

2nd case: long wavelength $k^2 \ll \theta''/\theta$

$$\theta u_{\mathbf{k}}'' - \theta'' u_{\mathbf{k}} = 0$$

$$u_{\mathbf{k}} \simeq C_1 \theta + C_2 \theta \int_{\eta_i} \frac{d\eta}{\theta^2},$$

$$= \frac{A}{\varphi'} \left[\frac{1}{a} \int_{\eta_i} a^2(\eta)\, d\eta\right]'$$

$$\Phi(t, \mathbf{x}) \simeq A \left(1 - \frac{H}{a} \int a\, dt\right)$$

Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

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2nd case: long wavelength $k^2 \ll \theta''/\theta$

$$\theta u_{\mathbf{k}}'' - \theta'' u_{\mathbf{k}} = 0$$

$$\Phi(t, \mathbf{x}) \simeq A\left(1 - \frac{H}{a} \int a\, dt\right)$$

Long-wavelength perturbations (outside the Hubble radius) are ***frozen***: don't grow while super-Hubble

Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

Scalar modes:

2nd case: long wavelength $k^2 \ll \theta''/\theta$

$$u_{\mathbf{k}}'' + \left(k^2 - \frac{\theta''}{\theta}\right) u_{\mathbf{k}} = 0$$

Gauge invariant conserved variable outside Hubble radius

Comoving curvature perturbation

$$\mathcal{R} \equiv \Phi + \frac{2\mathcal{H}}{8\pi G}\frac{1}{\varphi'^2}\left(\Phi' + \mathcal{H}\Phi\right)$$

It represents the gravitational potential in comoving surfaces $\delta\phi = 0$

$$\mathcal{R}' = \frac{2\mathcal{H}}{8\pi G}\frac{1}{\varphi'^2}\nabla^2\Phi \xrightarrow{\nabla^2\Phi \to 0} 0$$

Conserved!

Curvature perturbation

in a hypersurface of constant density

$$\zeta \equiv \Phi + \frac{2}{3}\frac{H^{-1}\dot{\Phi} + \Phi}{1+\omega}$$

$\nabla^2\Phi \to 0$

$$\dot{\zeta}\left(1+\omega\right) = 0$$

Conserved!

Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

Scalar modes:

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$$u''_{\mathbf{k}} + \left(k^2 - \frac{\theta''}{\theta}\right) u_{\mathbf{k}} = 0$$

Gauge invariant conserved variable outside Hubble radius

$$\mathcal{R}' = 0 \text{ and } \dot{\zeta} = 0$$

$$\implies \qquad \mathcal{R}\,,\zeta \to \text{const.}$$ on super-Hubble scales

Perturbations on these scales are frozen and remain like that until Hubble radius re-entering

Theory of cosmological perturbations in GR

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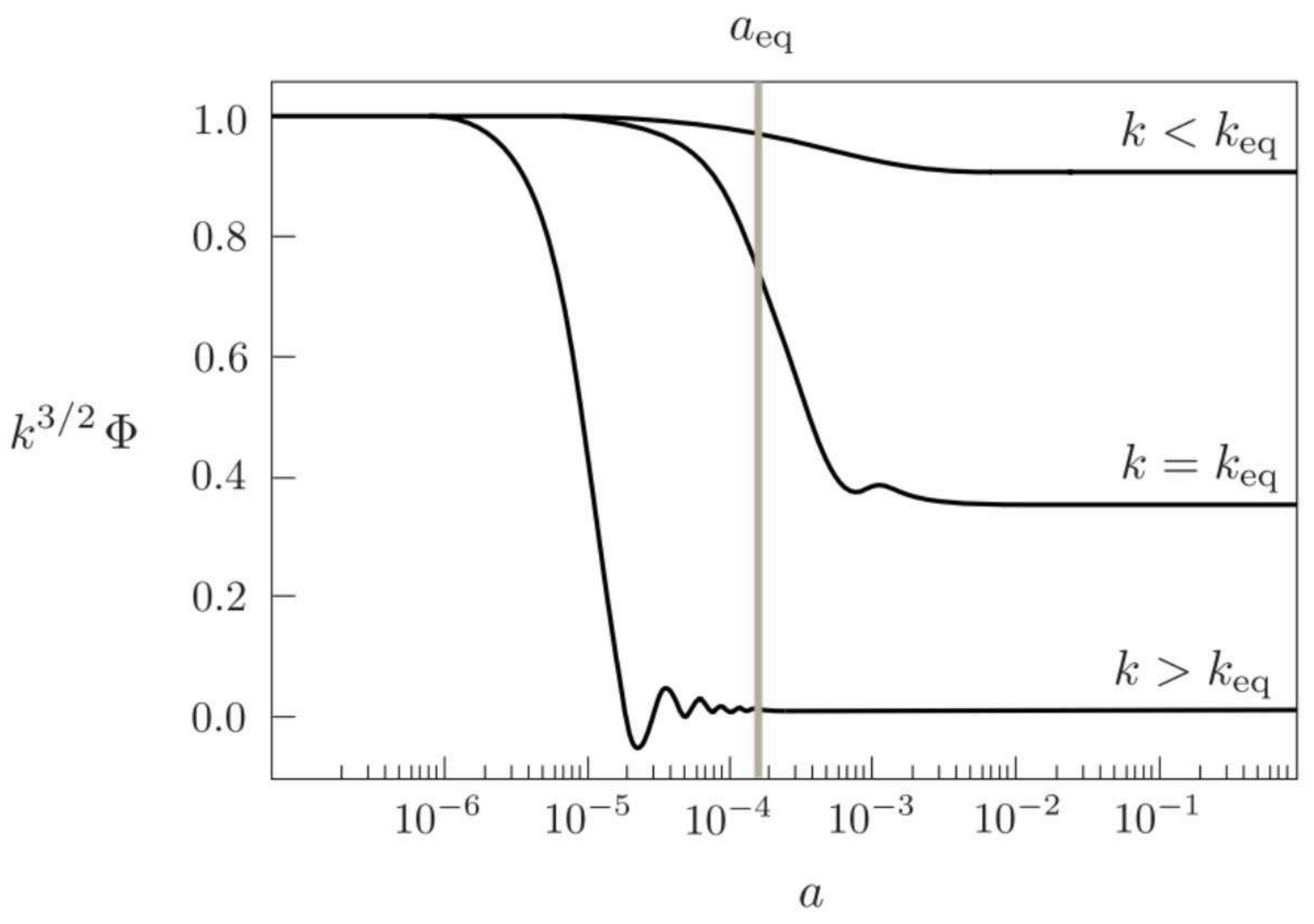
Slow-roll inflation: $\mathscr{R} = \zeta$

Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

Scalar modes:



Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

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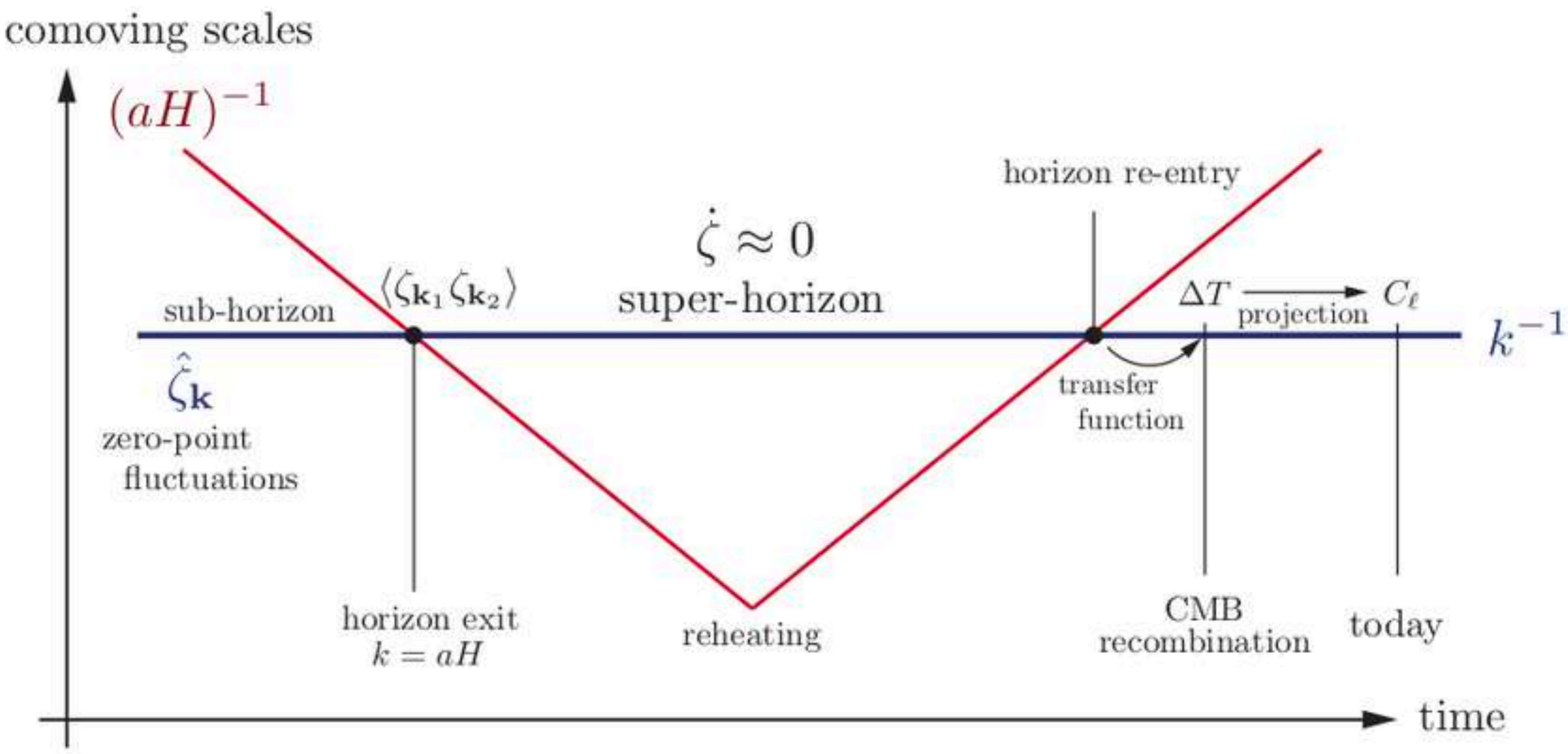

Theory of cosmological perturbations in GR

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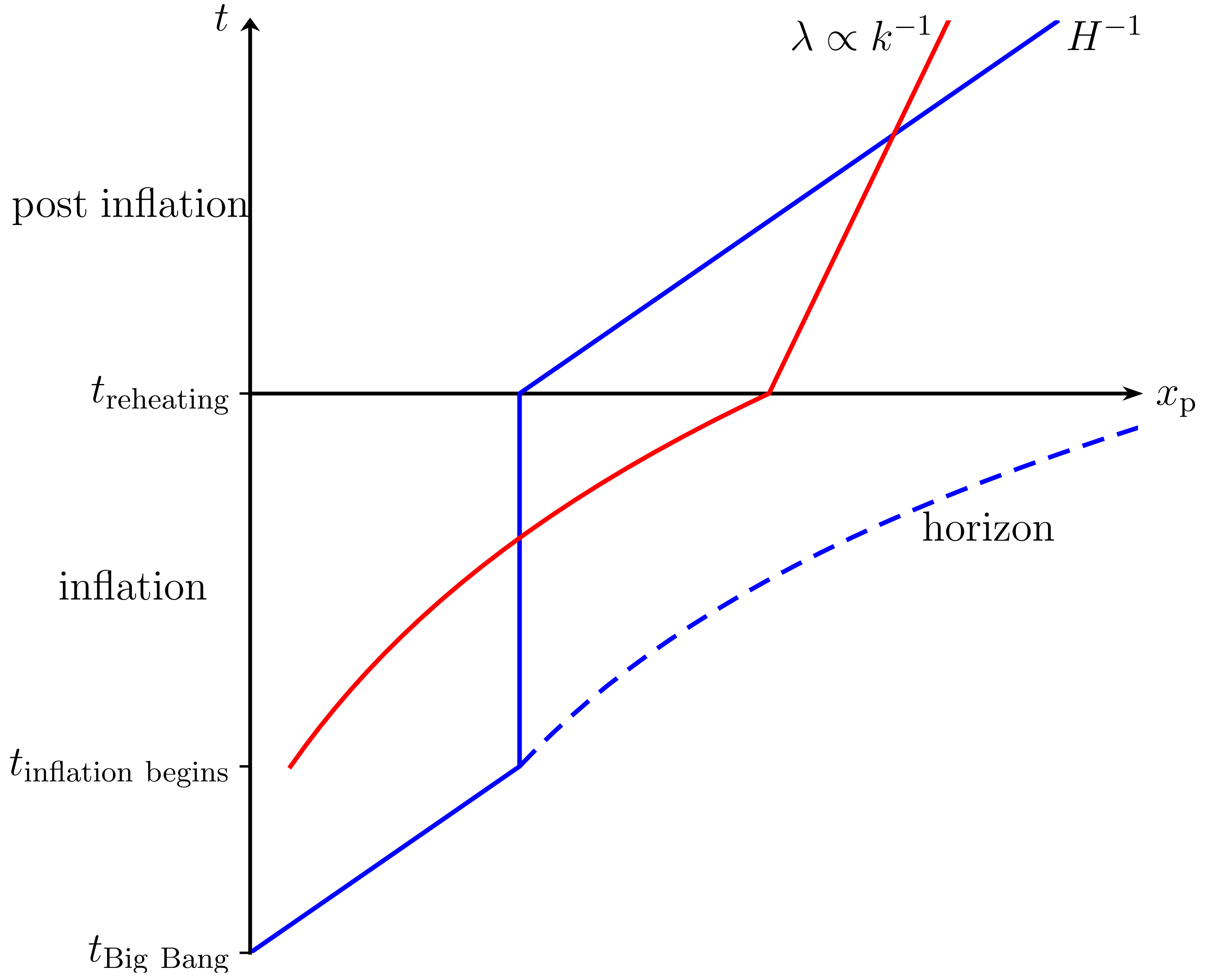

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Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

Vector modes: only metric perturbations; do not contain inflaton perturbations

$$\nabla^2 \Phi_i = 0$$

$$\Phi_i^{\prime} + 2\mathcal{H}\Phi_i = 0$$

Solution: $\Phi_i\left(\eta, \mathbf{x}\right) = C_i\left(\mathbf{x}\right) a^{-2}$

Decays fast as the universe expands!

For those modes to be relevant, they had to be ***large*** initially.

If the seed of perturbations is generated during inflation, they cannot be too large to be detected today

Theory of cosmological perturbations in GR

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The vector modes **do not** contribute to gravitational collapse and structure formation.

They represent **rotational** movements of the cosmic fluid

Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

Tensor modes: only metric perturbations; do not contain inflaton perturbations

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 0$$

Damped harmonic oscillator

Solution:

$$\mu''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right)\mu_{\mathbf{k}} = 0$$

$$h_{ij} = \frac{1}{a(\eta)} \int \frac{d^3k}{(2\pi)^3} \left[\mu_{\mathbf{k}}(\eta)\, \epsilon_{ij}(\mathbf{k})\, e^{i\mathbf{kx}} + \text{c.c.}\right]$$

$$\epsilon_{ij} = \epsilon_{ji}\,, \epsilon^i_j = \epsilon_{ij}k^i = 0$$

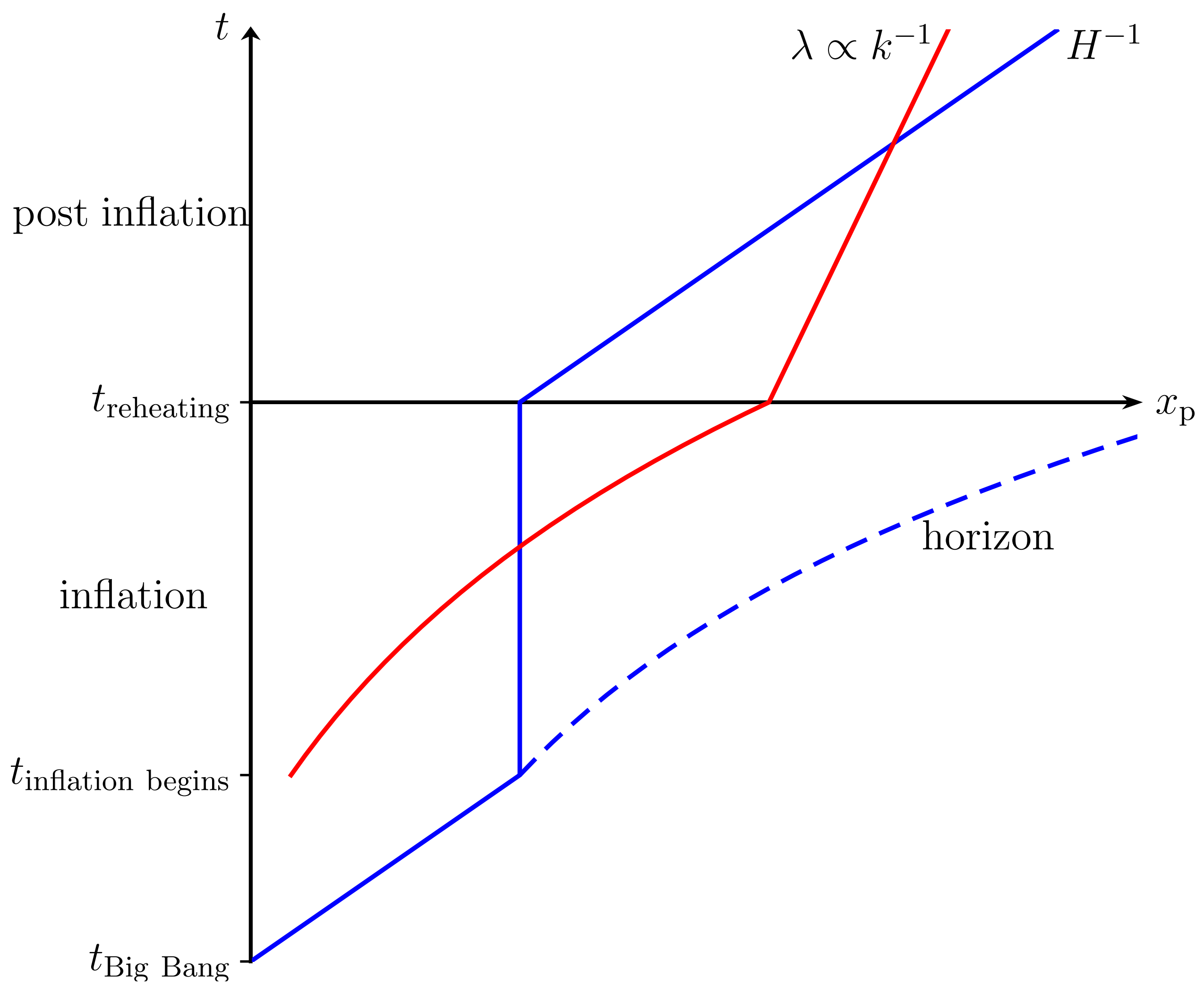
Transverse, traceless

The tensor modes **do not** contribute to gravitational collapse and structure formation.
BUT, they are very important since they represent the **gravitational waves!**

Theory of cosmological perturbations in GR

LINEARIZED EINSTEIN EQUATIONS

Scalar modes:



Vector modes:

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Decays fast as the universe expands!

Tensor modes:

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$$\epsilon_{ij} = \epsilon_{ji}\,, \epsilon^i_j = \epsilon_{ij}k^i = 0$$

Transverse, traceless

Gravitational waves!!

The ***scalar modes*** are the ONLY ones coupled to matter.
Relevant modes for structure formation!

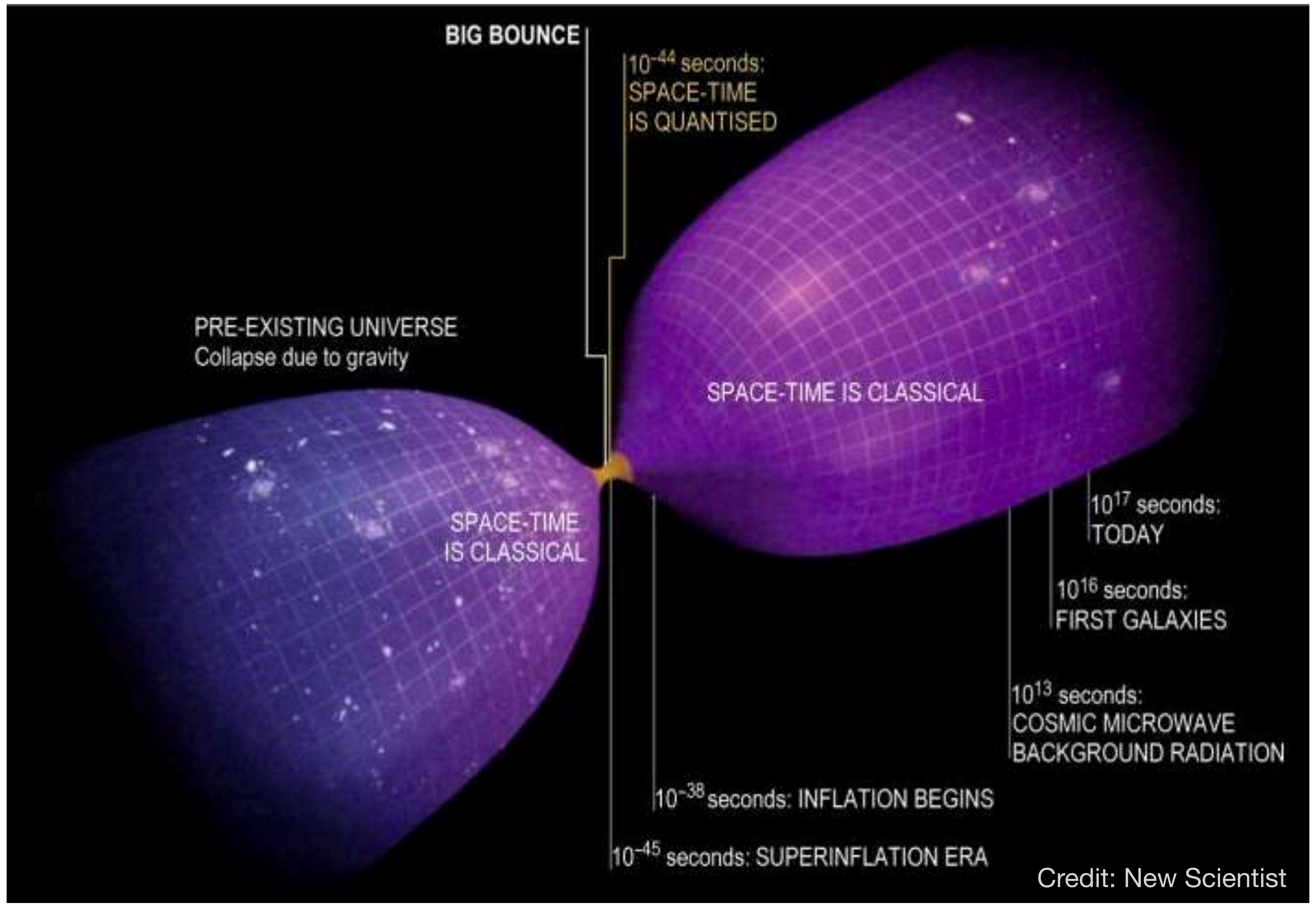
The vector and tensor modes **do not** contribute to gravitational collapse and structure formation.

Next lesson

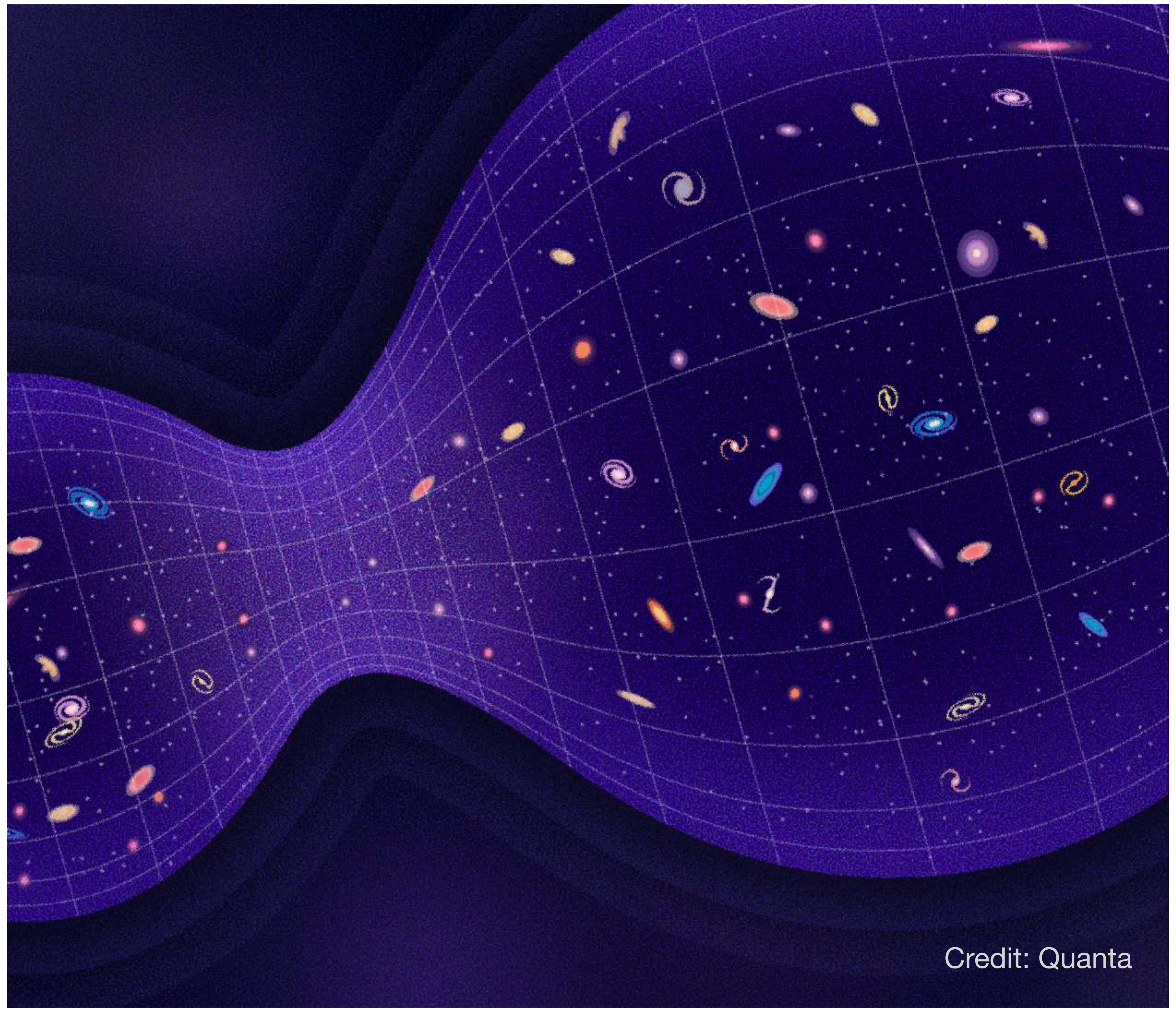
Quantum theory of cosmological perturbations

Observables

Alternatives to inflation



Credit: New Scientist


Credit: Quanta